THETA SPACES IN S^2

The purpose of this writeup is to give a proof of Lemma 64.1 in `planargraphs.pdf` (taken from Munkres, *Topology*) which only depends upon the results in this course and not upon Theorem 63.5 from the cited file.

Recall that a theta space is a union of three compact subspaces A, B, C which are homeomorphic to $[0, 1]$ such that the intersections $A \cap B$, $B \cap C$ and $A \cap C$ are the common endpoints. It follows that the union of any two of A, B, C is homeomorphic to S^1. Denote the set of common endpoints by $\{p, q\}$.

LEMMA 64.1. *Let $X \subset S^2$ be a theta space. Then $S^2 - X$ has three components whose frontiers are $A \cup B$, $B \cup C$ and $A \cup C$. The component having $A \cup B$ as its frontier is a connected component of $S^2 - (A \cup B)$.*

Homological proof. Let U and V be the connected components of $S^2 - (A \cup B)$, and let $C_0 = C - \{p, q\}$. Then by connectedness C_0 is contained in either U or V; without loss of generality, we might as well assume $C_0 \subset V$ (compare Figure 64.3 in `planargraphs.pdf`).

We also know that $\overline{U} = U \cup A \cup B = S^2 - V$ and that $S^2 - C$ is acyclic. Furthermore, the proof of the Jordan-Brouwer Theorem implies that both U and V are acyclic spaces. Since the union of $S^2 - \overline{U}$ and $S^2 - C$ subspaces is $S^2 - \{p, q\}$, the Mayer-Vietoris sequence for this triad of open subspaces in S^2 implies that the homology of the intersection

$$S^2 \ - \ (\overline{U} \ \cup \ C)$$

is isomorphic to the homology of S^2, so that the intersection has exactly two components, and each is acyclic. On the other hand, this intersection is also equal to $V - C_0$, and hence the latter also has exactly two components, each of which is acyclic. It follows that $S^2 - X = (V - C_0) \cup U$ has exactly three components, each of which is acyclic. Furthermore, one of these components is U, and the other two are contained in V.

Similarly, one component of $S^2 - X$ is a component of $S^2 - (B \cup C)$ while the other two components of $S^2 - X$ lie in a single component of $S^2 - (B \cup C)$, and one component of $S^2 - X$ is a component of $S^2 - (A \cup C)$ while the other two components of $S^2 - X$ lie in a single component of $S^2 - (A \cup C)$. These three components of $S^2 - X$ must all be different, for their frontiers are given by the three subsets $A \cup B$, $B \cup C$ and $A \cup C$ respectively (this point is glossed over in Munkres).

With this result at our disposal, the proof of Theorem 64.2 (the utilities graph is nonplanar) follows exactly as in `planargraphs.pdf`. The proof of Theorem 64.3 (the complete graph on five vertices is nonplanar) can be done similarly if we can eliminate the need to cite Theorem 63.5 in the proof of Lemma 64.3.

Using the notation and setting on page 397 of `planargraphs.pdf`, we may do this as follows: Each of the connected open subsets U, V, W is acyclic by the preceding discussion. The complement of a_2a_4 is connected by one step in the proof of the Jordan-Brouwer Theorem, and thus we can use the Mayer-Vietoris sequence for the open subset pair

$$\left(S^2 - (\overline{U} \cup \overline{V}), S^2 - a_2a_4\right)$$

(whose intersection is $S^2 - \{a_2, a_4\}$) to conclude that the complement of $\overline{U} \cup \overline{V} \cup a_2a_4$ has two components, each of which is acyclic. With these facts at our disposal, the remainder of the proof of Lemma 64.3 goes through as in `planargraphs.pdf`, and likewise the proof of Theorem 64.4 goes through unchanged.