Let a and b be points of W_1 and W_2, respectively. Because C_2 does not separate S^2, we can find a path α in $S^2 - C_2$ joining a and b. The set $\alpha(I)$ must contain a point y of the set $\bar{W}_1 - W_1$, because otherwise $\alpha(I)$ would be a connected set lying in the union of the disjoint open sets W_1 and $S^2 - \bar{W}_1$, and intersecting each of them. The point y belongs to the closed curve C, since $(\bar{W}_1 - W_1) \subset C$. Because the path α does not intersect the arc C_2, the point y must therefore lie in the arc C_1, which in turn lies in the open set U. Thus, U intersects $\bar{W}_1 - W_1$ in the point y, as desired. ■

Just as with the earlier theorems, we now ask ourselves what made the proof of this theorem work. Examining Step 1 of the proof, we see that all we used were the facts that C_1 and C_2 were closed connected sets, that $C_1 \cap C_2$ consisted of two points, and that neither C_1 nor C_2 separated S^2. The first two facts implied that $C_1 \cup C_2$ separated S^2 into at least two components; the third implied that there were *only* two components. Hence one has, with no further effort, the following result:

Theorem 63.5. *Let C_1 and C_2 be closed connected subsets of S^2 whose intersection consists of two points. If neither C_1 nor C_2 separates S^2, then $C_1 \cup C_2$ separates S^2 into precisely two components.*

EXAMPLE 1. The second half of the Jordan curve theorem, to the effect that C is the common boundary of W_1 and W_2, may seem so obvious as hardly to require comment. But it depends crucially on the fact that C is homeomorphic to S^1.

For instance, consider the space indicated in Figure 63.5. It is the union of two arcs whose intersection consists of two points, so it separates S^2 into two components W_1 and W_2 just as the circle does, by Theorem 63.5. But C does not equal the common boundary of W_1 and W_2 in this case.

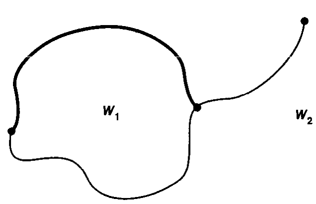

Figure 63.5

There is a fourth theorem that is often considered along with these three separation theorems. It is called the *Schoenflies theorem*, and it states that if C is a simple closed curve in S^2 and U and V are the components of $S^2 - C$, then $\bar{U}$ and $\bar{V}$ are each homeomorphic to the closed unit ball B^2. A proof may be found in [H-S].

The separation theorems can be generalized to higher dimensions as follows:

(1) Any subspace C of S^n homeomorphic to S^{n-1} separates S^n.

(2) No subspace A of S^n homeomorphic to $[0, 1]$ or to some ball B^m separates S^n.

(3) Any subspace C of S^n homeomorphic to S^{n-1} separates S^n into two components, of which C is the common boundary.

These theorems can be proved quite readily once one has studied singular homology groups in algebraic topology. (See [Mu], p. 202.) The Brouwer theorem on invariance of domain for $\mathbb{R}^n$ follows as a corollary.

The Schoenflies theorem, however, does not generalize to higher dimensions without some restrictions on the way the space C is imbedded in S^n. This is shown by the famous example of the "Alexander horned sphere," a homeomorphic image of S^2 in S^3, one of whose complementary domains is not simply connected! (See [H-Y], p. 176.)

The separation theorems can be generalized even further than this. The definitive theorem along these lines is the famous *Alexander-Pontryagin duality theorem*, a rather deep theorem of algebraic topology, which we shall not attempt to state here. (See [Mu].) It implies that if the closed subspace C separates S^n into k components, so does any subspace of S^n that is homeomorphic to C (or even homotopy equivalent to C). The separation theorems (1)–(3) are immediate corollaries.

Exercises

1. Let C_1 and C_2 be disjoint simple closed curves in S^2.

(a) Show that $S^2 - C_1 - C_2$ has precisely three components. [*Hint:* If W_1 is the component of $S^2 - C_1$ disjoint from C_2, and if W_2 is the component of $S^2 - C_2$ disjoint from C_1, show that $\bar{W}_1 \cup \bar{W}_2$ does not separate S^2.]

(b) Show that these three components have boundaries C_1 and C_2 and $C_1 \cup C_2$, respectively.

2. Let D be a closed connected subspace of S^2 that separates S^2 into n components.

(a) If A is an arc in S^2 whose intersection with D consists of one of its end points, show that $D \cup A$ separates S^2 into n components.

(b) If A is an arc in S^2 whose intersection with D consists of its end points, show that $D \cup A$ separates S^2 into $n + 1$ components.

(c) If C is a simple closed curve in S^2 that intersects D in a single point, show $D \cup C$ separates S^2 into $n + 1$ components.

***3.** (a) Let D be a subspace of S^2 homeomorphic to the topologist's sine curve $\bar{S}$. (See §24.) Show that D does not separate S^2. [*Hint:* Let $h : \bar{S} \to D$ be the homeomorphism. Given $0 < c < 1$, let $\bar{S}_c$ equal the intersection of $\bar{S}$ with the set $\{(x, y) \mid x \leq c\}$. Show that given $a, b \in S^2 - D$, there is, for some value of c, a path in $S^2 - h(\bar{S}_c)$ from a to b. Conclude that there is a path in $S^2 - D$ from a to b.]

(b) Let C be a subspace of S^2 homeomorphic to the closed topologist's sine curve. Show that C separates S^2 into precisely two components, of which C is the common boundary. [*Hint:* Let h be the homeomorphism of the closed topologist's sine curve with C. Let $C_0 = h(0 \times [-1, 1])$. Show first, using the argument of Theorem 63.4, that each point of $C - C_0$ lies in the boundary of each component of $S^2 - C$.]

§64 Imbedding Graphs in the Plane

A (finite) ***linear graph*** G is a Hausdorff space that is written as the union of finitely many arcs, each pair of which intersect in at most a common end point. The arcs are called the ***edges*** of the graph, and the end points of the arcs are called the ***vertices*** of the graph.

Linear graphs are used in mathematics to model many real-life phenomena; however, we shall look at them simply as interesting spaces that in some sense are generalizations of simple closed curves.

Note that any graph is determined completely (up to homeomorphism) by listing its vertices and specifying which pairs of vertices have an edge joining them.

EXAMPLE 1. If G contains exactly n vertices, and if for every pair of distinct vertices of G there is an edge of G joining them, then G is called the ***complete graph on n vertices*** and is denoted G_n. Several such graphs are pictured in Figure 64.1. Note that the first three of these graphs are pictured as subspaces of $\mathbb{R}^2$, but the fourth is pictured instead as a subspace of $\mathbb{R}^3$. A little experimentation will convince you that this graph *cannot* in fact be imbedded in $\mathbb{R}^2$. We shall prove this result shortly.

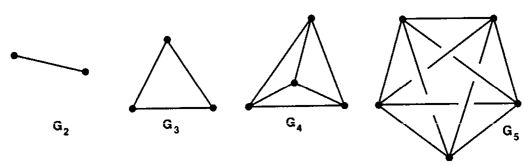

Figure 64.1

EXAMPLE 2. Another interesting graph arises in considering the classical puzzle: "Given three houses, h_1, h_2, and h_3, and three utilities, g (for gas), w (for water), and e (for electricity), can you connect each utility to each house without letting any of the connecting lines cross?" Formulated mathematically, this is just the question whether the graph pictured in Figure 64.2, which is called the ***utilities graph***, can be imbedded in $\mathbb{R}^2$. Again, a little experimentation will convince you that it cannot, a fact that we shall prove shortly.

Definition. A ***theta space*** X is a Hausdorff space that is written as the union of three arcs A, B, and C, each pair of which intersect precisely in their end points. (The space X is of course homeomorphic to the Greek letter theta.)

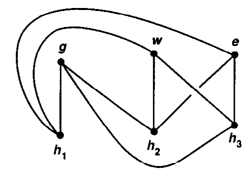

Figure 64.2

Note that as it stands, a theta space X is not a linear graph, for the arcs in question intersect in more than a common end point. One can write it as a graph, however, by breaking each of the arcs A, B, and C up into two arcs with an end point in common.

Lemma 64.1. *Let X be a theta space that is a subspace of S^2; let A, B, and C be the arcs whose union is X. Then X separates S^2 into three components, whose boundaries are $A \cup B$, $B \cup C$, and $A \cup C$, respectively. The component having $A \cup B$ as its boundary equals one of the components of $S^2 - A \cup B$.*

Proof. Let a and b be the end points of the arcs A, B, and C. Consider the simple closed curve $A \cup B$; it separates S^2 into two components U and U', each of which is open in S^2 and has boundary $A \cup B$. See Figure 64.3.

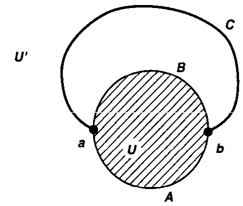

Figure 64.3

The space $C - a - b$ is connected, so it is contained in one of these components, say in U'. Then consider the two spaces $\bar{U} = U \cup A \cup B$ and C; each is connected. Neither separates S^2, for C is an arc, and the complement of $\bar{U}$ is the connected set U'. Since the intersection of these two sets consists of the two points a and b, their union separates S^2 into two components V and W, by Theorem 63.5. It follows that $S^2 - (A \cup B \cup C)$ is the union of the three disjoint connected sets U, V, and W; because they are open in S^2, they are the components of $S^2 - (A \cup B \cup C)$. The component U has $A \cup B$ as its boundary. Symmetry implies that the other two have $B \cup C$ and

$A \cup C$ as their boundaries. ∎

Theorem 64.2. *Let X be the utilities graph. Then X cannot be imbedded in the plane.*

Proof. If X can be imbedded in the plane, then it can be imbedded in S^2. So suppose X is a subspace of S^2. We derive a contradiction.

We use the notation of Example 2, where g, w, e, h_1, h_2, and h_3 are the vertices of X. Let A, B, and C be the following arcs contained in X:

$$A = gh_1w,$$
$$B = gh_2w,$$
$$C = gh_3w.$$

Each pair of these arcs intersect in their end points g and w alone; hence $Y = A \cup B \cup C$ is a theta space. The space Y separates S^2 into three components U, V, and W, whose boundaries are $A \cup B$, $B \cup C$, and $A \cup C$, respectively. See Figure 64.4.

Now the vertex e of X lies in one of these three components, so that the arcs eh_1 and eh_2 and eh_3 of X lie in the closure of that component. That component cannot be U, for $\bar{U}$ is contained in $U \cup A \cup B$, a set that does not contain the point h_3. Similarly, the component containing e cannot be V or W, because $\bar{V}$ does not contain h_1, and $\bar{W}$ does not contain h_2. Thus, we have reached a contradiction. ∎

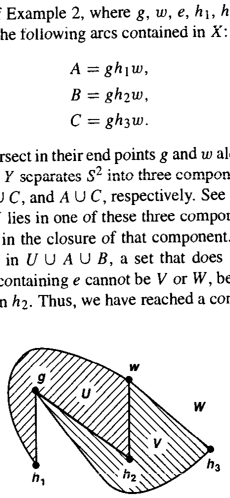

Figure 64.4

Lemma 64.3. *Let X be a subspace of S^2 that is a complete graph on four vertices a_1, a_2, a_3, and a_4. Then X separates S^2 into four components. The boundaries of these components are the sets X_1, X_2, X_3, and X_4, where X_i is the union of those edges of X that do not have a_i as a vertex.*

Proof. Let Y be the union of all the arcs of X different from the arc a_2a_4. Then we can write Y as a theta space by setting

$$A = a_1a_2a_3,$$
$$B = a_1a_3,$$
$$C = a_1a_4a_3.$$

See Figure 64.5. The arcs A, B, and C intersect in their end points a_1 and a_3 alone, and their union is Y.

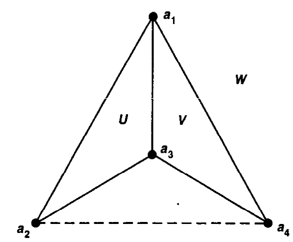

Figure 64.5

The space Y separates S^2 into three components U, V, and W, whose boundaries are $A \cup B$, $B \cup C$, and $A \cup C$, respectively. The space $a_2a_4 - a_2 - a_4$, being connected, must lie in one of them. It cannot lie in U, because $A \cup B$ does not contain a_4. And it cannot lie in V because $B \cup C$ does not contain a_2. Hence it must lie in W.

Now $\bar{U} \cup \bar{V}$ is connected because $\bar{U}$ and $\bar{V}$ are connected and have nonempty intersection B. Furthermore, the set $\bar{U} \cup \bar{V}$ does not separate S^2, because its complement is W. Similarly, the arc a_2a_4 is connected and does not separate S^2. And the sets a_2a_4 and $\bar{U} \cup \bar{V}$ intersect in the points a_2 and a_4 alone. It follows from Theorem 63.5 that $a_2a_4 \cup \bar{U} \cup \bar{V}$ separates S^2 into two components W_1 and W_2. Then $S^2 - Y$ is the union of the four disjoint connected sets U, V, W_1, and W_2. Since these sets are open, they are the components of $S^2 - Y$.

Now one of these components, namely U, has the graph $A \cup B = X_4$ as its boundary. Symmetry implies that the other three have X_1, X_2, and X_3 as their respective boundaries. ∎

Theorem 64.4. *The complete graph on five vertices cannot be imbedded in the plane.*

Proof. Suppose that G is a subspace of S^2 that is a complete graph on the five vertices a_1, a_2, a_3, a_4, and a_5. Let X be the union of those edges of G that do not have a_5 as a vertex; then X is a complete graph on four vertices. The space X separates S^2 into four components, whose respective boundaries are the graphs $X_1, \ldots, X_4$, where X_i consists of those edges of X that do not have a_i as a vertex. Now the point a_5 must lie in one of these four components. It follows that the connected space

$$a_1a_5 \cup a_2a_5 \cup a_3a_5 \cup a_4a_5,$$

which is the union of those edges of G that have a_5 as a vertex, must lie in the closure of this component. Then all the vertices $a_1, \ldots, a_4$ lie in the boundary of this component.

But this is impossible, for none of the graphs X_i contains all four vertices $a_1, \dots, a_4$. Thus we reach a contradiction. ■

It follows from these theorems that if a graph G contains a subgraph that is a utilities graph or a complete graph on five vertices, then G cannot be imbedded in the plane. It is a remarkable theorem, due to Kuratowski, that the converse is also true! The proof is not easy.

Exercise

1. Let X be a space that is written as the union of finitely many arcs $A_1, \dots, A_n$, each pair of which intersect in at most a common end point.
   (a) Show that X is Hausdorff if and only if each arc A_i is closed in X.
   (b) Give an example to show that X need not be Hausdorff. [*Hint:* See Exercise 5 of §36.]

§65 The Winding Number of a Simple Closed Curve

If $h : S^1 \to \mathbb{R}^2 - \mathbf{0}$ is a continuous map, then the induced homomorphism h_* carries a generator of the fundamental group of S^1 to some integral power of a generator of the fundamental group of $\mathbb{R}^2 - \mathbf{0}$. This integral power n is called the *winding number* of h with respect to $\mathbf{0}$. It measures how many times h "wraps S^1 around the origin;" its sign depends of course on the choice of generators. See Figure 65.1. We will introduce it more formally in the next section.

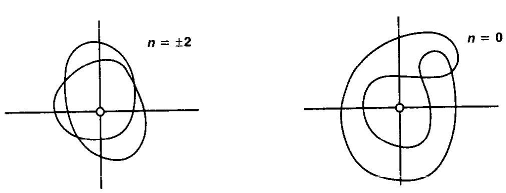

Figure 65.1

For the present, we merely ask the question: What can one say about the winding number of h if h is injective, that is, if h is a homeomorphism of S^1 with a simple closed curve C in $\mathbb{R}^2 - \mathbf{0}$? The illustrations in Figure 65.2 suggest the obvious conjecture: If $\mathbf{0}$ belongs to the unbounded component of $\mathbb{R}^2 - C$, then $n = 0$, while if $\mathbf{0}$ belongs to the bounded component, then $n = \pm 1$.

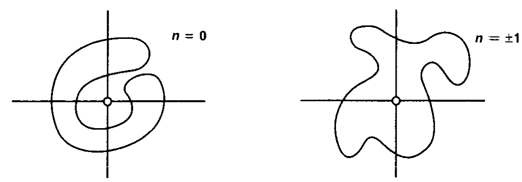

Figure 65.2

The first conjecture is easy to prove, for Lemma 61.2 tells us that h is nulhomotopic if $\mathbf{0}$ belongs to the unbounded component of $\mathbb{R}^2 - C$. On the other hand, the second conjecture is surprisingly difficult; it is in fact a rather deep result. We prove it in this section.

As usual, we shall replace $\mathbb{R}^2 \cup \{\infty\}$ by S^2, letting p be the point corresponding to $\mathbf{0}$ and q be the point corresponding to ∞. Then our conjecture can be reformulated as follows: If C is a simple closed curve in S^2, and if p and q belong to different components of $S^2 - C$, then the inclusion mapping $j : C \to S^2 - p - q$ induces an isomorphism of fundamental groups. This is what we shall prove.

First, we prove our result in the case where the simple closed curve C is contained in a complete graph on four vertices. Then we prove the general case.

Lemma 65.1. *Let G be a subspace of S^2 that is a complete graph on four vertices $a_1, \dots, a_4$. Let C be the subgraph $a_1a_2a_3a_4a_1$, which is a simple closed curve. Let p and q be interior points of the edges a_1a_3 and a_2a_4, respectively. Then:*

(a) The points p and q lie in different components of $S^2 - C$.

(b) The inclusion $j : C \to S^2 - p - q$ induces an isomorphism of fundamental groups.

Proof. (a) As in the proof of Lemma 64.3, the theta space $C \cup a_1a_3$ separates S^2 into three components U, V, and W. One of these, say W, has C as its boundary; it is the only component whose boundary contains both a_2 and a_4. Therefore, $a_2a_4 - a_2 - a_4$ must lie in W, so that in particular, q belongs to W. Of course, p is not in W because p belongs to the theta space $C \cup a_1a_3$. Now Lemma 64.1 tells us that W is one of the components of $S^2 - C$; therefore, p and q belong to different components of $S^2 - C$.

(b) Let $X = S^2 - p - q$. The idea of the proof is the following: We choose a point x interior to the arc a_1a_2, and a point y interior to the arc a_3a_4. And we let α and β be the broken-line paths

$$\alpha = xa_1a_4y \quad \text{and} \quad \beta = ya_3a_2x.$$

Then $\alpha * \beta$ is a loop lying in the simple closed curve C. We shall prove that $\alpha * \beta$ represents a generator of the fundamental group of X. It follows that the homomorphism