Prismatoids and their volumes

A ***convex linear cell*** (sometimes also called a ***convex polytope***) is a closed **bounded** subset of some $\mathbb{R}^n$ defined by a finite number of linear equations or inequalities. Note that sets defined by finite systems of this type are automatically convex. Prisms, pyramids (including tetrahedra) and cubes are obvious examples, but of course there are also many others including regular Platonic solids which are octahedral, dodecahedra and icosahedra. For every such object, there is a finite set $E = \{e_1, \dots\}$ of ***extreme points*** or ***vertices*** such that the cell is the set of all (finite) convex combinations of the extreme points; in other words, for each x in the cell there are scalars t_j such that $t_j \geq 0$,

$$\sum_j t_j = 1$$

and

$$x = \sum_j t_j e_j .$$

Such a cell is said to be n – ***dimensional*** if it is not contained in any hyperplane; this is equivalent to assuming the set has a nonempty interior. In some sense the least complicated examples of this sort is an n – ***simplex***, for which the set of vertices consists of $n + 1$ points $\{v_1, \dots, v_n\}$ such that the vectors $v_i - v_0$ (where $i > 0$) are linearly independent; if $n = 2$ this corresponds to a triangle, and if $n = 3$ it corresponds to a pyramid with a triangular base (a ***tetrahedron***). Further information on simplices and spaces built from them appears on pages 15 – 16 of the following online reference:

http://math.ucr.edu/~res/math246A/algtopnotes2009.pdf

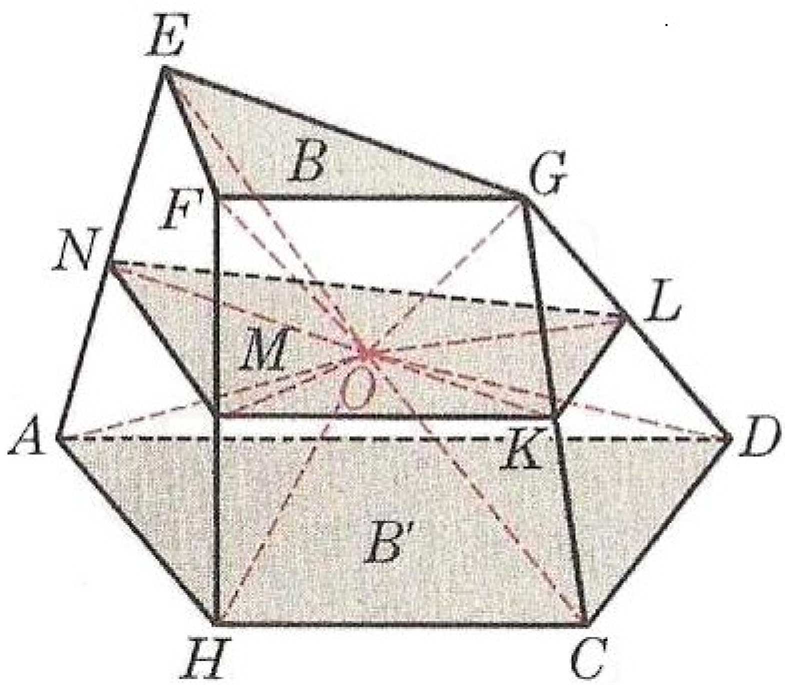
A basic theorem on convex sets states that *every convex linear cell has a **simplicial decomposition** for which E is the set of vertices.* — Proofs of this statement appear in the books by Munkres and Hudson cited below; specific references are Section 7 of Munkres and pages 1 – 14 of the book by Hudson.

J. F. P. Hudson. ***Piecewise Linear Topology*** (Notes by J. Shaneson and J. A. Lees). W. A. Benjamin, New York, 1969.

J. R. Munkres. ***Elementary differential topology*** (Lectures given at MIT, Fall, 1961. Revised Edition), Annals of Mathematics Studies No. 54. Princeton University Press, Princeton, 1966.

In this document we are mainly interested in a special class of 3 – dimensional convex linear cells in $\mathbb{R}^3$ known as ***prismatoids***. The defining condition for such objects is that all their vertices must lie on a pair of parallel planes. Usually we shall assume that these two planes are the horizontal xy – plane defined by $z = 0$ and the parallel plane defined by $z = h$ for some $h > 0$.

Prisms and pyramids are examples of prismatoids, but the drawing below shows that there are also other examples:



This drawing is taken from pages 274 – 275 of the following textbook:

> **A. M. Welchons, W. R. Krickenberger and H. R. Pearson. *Solid Geometry*.** Ginn & Co., Boston, 1959.

Two additional online references with illustrations are given below; the second one has an interactive figure that can be rotated.

http://en.wikipedia.org/wiki/Prismatoid

http://mathworld.wolfram.com/Prismatoid.html

The main result

The following volume formula was formulated by Ernst Ferdinand August (1795 – 1870; more widely known for numerous contributions to physics such as measurement of relative humidity) near the middle of the 19th century:

Prismatoid Theorem. *Suppose that* **P** *is a prismatoid in* $\mathbf{3}$ *– space whose vertices lie in the two parallel planes* $z = 0$ *and* $z = h$. *Then the volume of* **P** *is given by the formula*

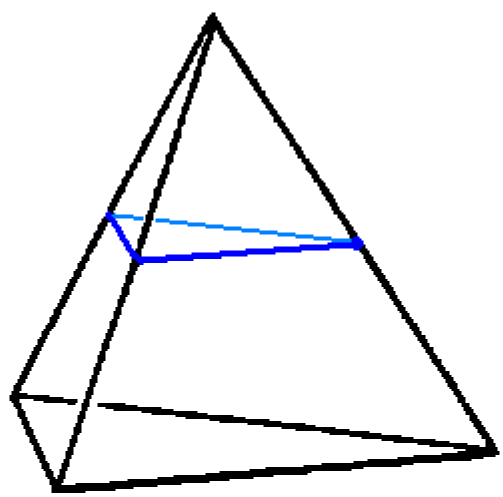
$$V = \frac{h}{6} \cdot (U + 4M + L)$$

where U *is the area of the upper convex linear cell obtained by intersecting* **P** *with the plane* $z = h$, L *is the area of the lower convex linear cell obtained by intersecting* **P** *with the plane* $z = 0$, *and* M *is the area of the midsection convex linear cell obtained by intersecting* **P** *with the plane* $z = h/2$.

Steps in the proof. The main idea is to reduce the proof to the case of a single simplex. We know that **P** has a simplicial decomposition with no extra vertices, so the volume of **P** is equal to the volumes of the individual pieces. Likewise, the areas of the upper, lower and midsection cells will be the sums of the corresponding cells for the individual simplices, and therefore the general case will follow by adding together the results for the various simplices. For each simplex there are two cases, depending upon how many of the four vertices lie on each of the two parallel planes: Specifically, in the first case one of the planes contains only one vertex and the other contains the other three, and in the second case each plane contains exactly two.

The first case

Suppose that we have a simplex such that three vertices lie on the plane $z = 0$ and one lies on the plane $z = h$, or vice versa. We shall consider the first of these alternatives; the second follows from similar considerations. As usual, let h be the altitude and let B be the area of the base.



Standard formulas for pyramid volumes imply that the volume of the simplex equals $Bh/3$. There is no upper face, and the area M of the midsection, which is similar to the base with similitude ratio $\frac{1}{2}$, is equal to $B/4$. Therefore the right hand side of the volume formula is equal to

$$\frac{h}{6} \cdot (B + 4M) = \frac{h}{6} \cdot \left(B + 4\left(\frac{B}{4}\right)\right) = \frac{Bh}{3}.$$

The second case

Assume now that each of the two parallel planes contains exactly two vertices. We shall need the following two formulas:

Formula 1. *Suppose that we are given a* $3-$ *simplex* **A** *in* $\mathbb{R}^3$ *whose vertices are* $x_1,\ x_2,$ x_3 *and* x_4. *Then the volume of the simplex* **A** *is given by the following formula*:

$$V \quad = \quad |\det(y_2 - y_1 \quad y_2 - y_1 \quad y_2 - y_1)|/6$$

As usual, we assume that the columns of the 3×3 matrix in this expression are given by the three listed vectors.

Derivation. Let **T** be the affine transformation on $\mathbb{R}^3$ sending (u, v, w) to

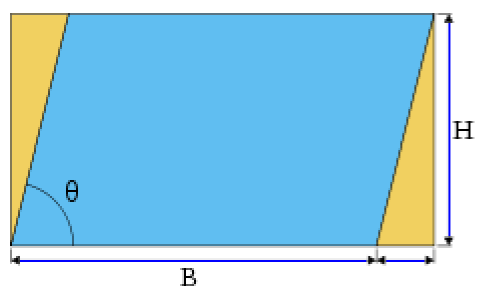
$$(1-u-v-w)x_1 + \ ux_2 + \ vx_3 + \ wx_4\,.$$

If **S** is the standard simplex in $\mathbb{R}^3$ whose vertices are the origin and the three standard unit vectors, then **T** maps **S** to **A**, and therefore by the usual Change of Variables Theorems the volume of **A** is equal to the volume of **S**, which is equal to $1/6$, times the absolute value of the Jacobian of **T**, which is just the cofactor of $1/6$ in the displayed formula. Therefore it follows that the volume of **A** is given by the displayed expression.

Formula 2. *Suppose that we are given vectors* a, b, c, d *in* $\mathbb{R}^3$ *which (in the given order) are the vertices of a parallelogram. Then the area of the region bounded by this parallelogram is given by the following formula*:

$$A \quad = \quad |\,(b-a) \times (d-a)\,|$$

Derivation. If θ denotes the angle between $b - a$ and $d - a$, then the altitude from d to the line ab has length equal to $|\,d-a\,|\sin\theta$.



(Source: http://en.wikipedia.org/wiki/File:Parallelogram_area.svg)

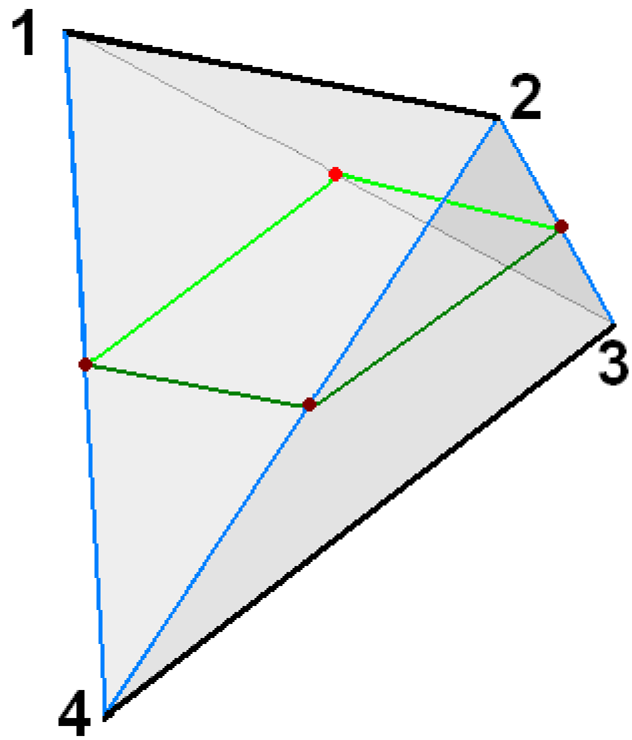
Therefore we have

$$A \quad = \quad |\,b-a\,|\,|\,d-a\,|\sin\theta \quad = \quad |\,(b-a)\times(d-a)\,|.$$

Proof of the formula in the second case. Motivated by the drawing below, take the vertices to be

$$x_1 \ = \ (y_1, h), \quad x_2 \ = \ (y_2, h), \quad x_3 \ = \ (y_3, 0)\,, \quad x_4 \ = \ (y_4, 0)$$

where the y_i are suitable $2-$dimensional vectors; since the x_i are not coplanar, it follows that $y_2 - y_1$ and $y_4 - y_3$ are linearly independent. The midsection of the simplex is its intersection with the hyperplane $z = h/2$, and as suggested in the drawing below it is the convex hull of the midpoints of the four edges in the simplex.



Specifically, the vertices of the midsection are given as follows:

$$a = \frac{1}{2}(y_1 + y_4, h), \qquad b = \frac{1}{2}(y_2 + y_4, h)$$

$$c = \frac{1}{2}(y_2 + y_3, h), \qquad d = \frac{1}{2}(y_1 + y_3, h)$$

In particular, it follows that $b - a = c - d$ and $d - a = c - b$, and therefore a, b, c, d (in the given order) form the vertices of a parallelogram, which means that the midsection is bounded by a parallelogram. Therefore by an earlier formula the area of the midsection is given by following expression:

$$M = \frac{1}{4} \cdot |(y_2 - y_1, 0) \times (y_3 - y_4, 0)|$$

If we now let $y_i = (u_i, v_i)$ then we obtain the following formula for the volume of the simplex:

$$V = \frac{1}{6} \cdot \left| \det \begin{pmatrix} u_2 - u_1 & u_3 - u_1 & u_4 - u_1 \\ v_2 - v_1 & v_3 - v_1 & v_4 - v_1 \\ 0 & -h & -h \end{pmatrix} \right|$$

We can now use the standard formulas for determinants to rewrite this formula as follows:

$$V = \frac{1}{6} \cdot \left| \det \begin{pmatrix} u_2 - u_1 & u_3 - u_4 & u_4 - u_1 \\ v_2 - v_1 & v_3 - v_4 & v_4 - v_1 \\ 0 & 0 & -h \end{pmatrix} \right|$$

We can now use the determinant formula for cross products to express the right hand side in terms of the latter:

$$V = \frac{h}{6} \cdot |\,(y_2 - y_1) \times (y_3 - y_4)|$$

$$= \frac{h}{6} \cdot |\,(2b - 2a, 0) \times (2d - 2a, 0)| =$$

$$\frac{h}{6} \cdot 4 \cdot |\,(b - a, 0) \times (d - a, 0)| = \frac{h}{6} \cdot 4\,M$$

Since there are two vertices in the top and bottom planes, it follows that the base areas are both zero and therefore in this case we also have

$$V = \frac{h}{6} \cdot (L + 4M + U)\,.$$

General reference for solid geometry

G. B. Halsted. ***Rational Geometry: A textbook for the Science of Space. Based on Hilbert's Foundations*** (Second Edition)**.** John Wiley and Sons, New York, 1907**.**