Relative cup products

The following material corrects the discussion of relative cup products in the notes.

The definition of the relative cup product pairing

$$H^*(X,A) \times H^*(X,B) \longrightarrow H^*(X, A\cup B)$$

(with coefficients in some commutative ring with unit) should include the assumption that A and B are open in X. The definition can be extended to other cases, but this requires additional work and the open case will suffice for the applications (in which subsets should be assumed to be open if necessary).

Here is a sketch of the justification, in which we assume that all cohomology coefficients are given by a fixed commutative ring with unit:

Suppose that A and B are open in X, let $S_*^{\mathcal{F}}(A\cup B)$ be the subcomplex of $\mathcal{F}$-small singular chains, let $S^*_{\mathcal{F}}(A\cup B)$ be the associated cochain complex, and let $S^*_{\mathcal{F}}(X, A\cup B)$ be the kernel of the restriction map from $S^*(X)$ to $S^*_{\mathcal{F}}(A\cup B)$. Equivalently, $S^*_{\mathcal{F}}(X, A\cup B)$ is the cochain complex associated to the quotient

$$S_*(X)/S_*^{\mathcal{F}}(A\cup B) \quad = \quad S_*(X)/\,(\,S_*(A)+S_*(B)\,)$$

and since A and B are open in X, it follows that $S^*_{\mathcal{F}}(X, A\cup B)$ is a quotient of $S^*(X,A)$ such that the projection from $S^*(X, A\cup B) \to S^*_{\mathcal{F}}(X, A\cup B)$ induces isomorphisms in cohomology.

Suppose now that we are given cochains $f \in S^p(X,A)$ and $g \in S^q(X,B)$; by construction $S^p(X,A)$ and $S^q(X,B)$ are cochain subcomplexes of $S^*(X)$, and therefore the cup product construction defines a cochain $f\cup g : S_{p+q}(X) \to \mathbb{D}$. We need to show that this cochain actually lies inside $S^*_{\mathcal{F}}(A\cup B)$, or equivalently that the restriction of $f\cup g$ to $S_*^{\mathcal{F}}(A\cup B) = S_*(A)+S_*(B)$ is trivial. This will follow if we can show that the restrictions of $f\cup g$ to both $S_*(A)$ and $S_*(B)$ are zero, and thus it suffices to show that $f\cup g(T) = 0$ if T is a singular simplex in A or B.

Suppose now that we are given a singular simplex T in A or B; symmetry considerations show it suffices to consider the first case (reverse the roles of the variables to get the other case). Then $f\cup g(T) = f(T_1)\cdot g(T_2)$, where T_i is obtained by restricting T to a front or back face of Δ_{p+q}. If the restriction of f to $S_*(A)$ is zero, then it follows from the previous formula that $f\cup g(T) = 0$. Similarly, if the restriction of g to $S_*(B)$ is zero, then one obtains the same conclusion. Therefore $f\cup g$ actually lies in $S^*_{\mathcal{F}}(A\cup B)$; the previous arguments show that $f\cup g$ is a cocycle if f and g are cocycles and in this case the cohomology class of $f\cup g$ depends only on the cohomology classes of f and g. This gives us a map from $H^p(X,A)\times H^q(X,B)$ to the cohomology of $S^*_{\mathcal{F}}(X, A\cup B)$, and since the surjection from $S^*(X, A\cup B)$ to this group induces cohomology isomorphisms it follows that we obtain a class in $H^*(X, A\cup B)$.■