Boundary maps for simplicial chains

Here is a carefully written out verification that $d_{k-1} \circ d_k = 0$.

Recall the definition

$$d_k (v_0 \dots v_k) = \sum_{i=0}^{k} (-1)^i v_0 \dots \hat{v_i} \dots v_k$$

$$= \sum_{i=0}^{k} (-1)^i \partial_i (v_0 \dots v_k)$$

Lemma 1, p. 51, in the notes

$$k-1 \geq j \geq i \Rightarrow \partial_j \partial_i = \partial_i \partial_{j+1}.$$

This is the key fact in proving $d_{k-1} \circ d_k = 0$.

To prove $d_{k-1} d_k (v_0 \dots v_k) = 0$ start out with the definitions:

$$d_{k-1} d_k (v_0 \dots v_k) = d_{k-1}\left(\sum_{i=0}^{k} (-1)^i \partial_i v_0 \dots v_k\right) =$$

$$\sum_{j=0}^{k-1} (-1)^j \partial_j \left(\sum_{i=0}^{k} (-1)^i \partial_i v_0 \dots v_k\right) =$$

$$\sum_{j=0}^{k-1} \sum_{i=0}^{k} (-1)^{i+j} \partial_j \partial_i v_0 \dots v_k =$$

$$\sum_{j \geq i} (-1)^{i+j} \partial_j \partial_i + \sum_{j<i} (-1)^{i+j} \partial_j \partial_i =$$

$$\sum_{j \geq i} (-1)^{i+j} \partial_i \partial_{j+1} + \sum_{j<i} (-1)^{i+j} \partial_j \partial_i .$$

Let's look at the first term more carefully, and make the change of variables $J = i$, $I = j+1$. Then this term translates into

$$\sum_{J<I} (-1)^{I+J+1} \partial_J \partial_I$$

the sum ranging over all I, J such that $J < I$, $0 \leq J \leq k$, $1 \leq I \leq k$.

Note that we could write the range of the summation as all I, J such that

(a) $0 \leq J \leq k-1$ because $J < I \Rightarrow J < k$

(b) $0 \leq I \leq k$ because $J < I \Rightarrow I \geq 1$.

Thus the entire original sum becomes

$$\sum_{J<I} (-1)^{I+J+1} \partial_J \partial_I + \sum_{j<i} (-1)^{i+j+1} \partial_i \partial_j$$

which is zero, as claimed.