EXCISION THEOREMS IN SINGULAR HOMOLOGY

The main result on excision in singular homology is the following, and its proof is basically given in the course notes:

THEOREM. *Let X be a topological space such that $X = B \cup C$, where the interiors of B and C cover X. Then the inclusion map from $(B, B \cap C)$ to $(X = B \cup C, C)$ induces isomorphisms from $H_*(B, B \cap C)$ to $H_*(B \cup C, C)$.*

Two special cases of this result are particularly important.

COROLLARY 1. *If (X, A) is a pair of topological spaces and U is a subset (not necessarily open!) such that $\overline{U} \subset \mathrm{Interior}(A)$, then the inclusion map the inclusion map from $(X - U, A - U)$ to (X, A) induces isomorphisms from $H_*(X - U, A - U)$ to $H_*(X, A)$.*

To derive this from the main theorem, let $B = X - U$ and $C = A$, so that $(B, B \cap C) = (X - U, A - U)$ and $(B \cup C, C) = (X, A)$. The interiors of B and C cover X because $X - \overline{U}$ is an open subset of $X - U = B$, and $\overline{U} \subset \mathrm{Interior}(A)$ implies that

$$X - \overline{U} \ \cup \ \mathrm{Interior}(A) \quad = \quad X \ .$$

The second consequence follows immediately from the theorem.

COROLLARY 2. *Let X be a topological space such that $X = U \cup V$, where U and V are open in X. Then the inclusion map from $(U, U \cap V)$ to $(X = U \cup V, V)$ induces isomorphisms from $H_*(U, U \cap V)$ to $H_*(U \cup V, V)$.*