REGULAR FUNDAMENTAL DOMAINS FOR CLOSED ORIENTED SURFACES

Let Σ be a closed, oriented (smooth) surface of genus $g \geq 2$. Basic results of surface theory imply that Σ has a riemannian metric of constant negative gaussian curvature, and we can rescale the metric so that this curvature is always -1. It also follows that the universal covering of Σ is also a complete riemannian manifold with a metric whose gaussian curvature is also equal to -1 at each point, and therefore the universal covering is isometric to the hyperbolic plane $\mathbb{H}^2$.

It is well known that Σ is homeomorphic to a $4g-$gon where the edges are identified in a suitable fashion. In fact, we can say more: Let $\Gamma = \pi_1(\Sigma, \text{basepoint})$ act on $\mathbb{H}^2$ by covering transformations so that $\Sigma \cong \mathbb{H}^2/\Gamma$. Then there is a **fundamental domain** D for this action which is the closed region bounded by a regular $4g-$gon. In other words, the composite map

$$D \;\subset\; \mathbb{H}^2 \;\longrightarrow\; \mathbb{H}^2/\Gamma \;=\; \Sigma$$

is onto, it is 1–1 away from the boundary, and on the boundary it agrees with the usual identification of edges which yields Σ. Regularity means that all edges have equal length and all vertex angles have equal measures. Furthermore, there is a **regular tessellation** of $\mathbb{H}^2$ by $4g-$gons of the given type. If $g = 1$, the comparable structure is the set of all closed square regions of the form

$$[m, m+1] \;\times\; [n, n+1]$$

where m and n are integers; note that the corresponding fundamental domain in this case is the solid square region $[0,1]\times[0,1]$, and every square in the decomposition of $\mathbb{R}^2 \;=\;$ universal covering (T^2) is a translate of the fundamental domain by a point in $\mathbb{Z}^2 \cong \pi_1(T^2, \text{basepoint})$. In the case of surfaces with genus ≥ 2, the tessellation presents $\mathbb{H}^2$ as a union of translates of D such that any two intersect in either a common vertex or a common edge, and there is some integer $k \geq 3$ such that there are k translates of D which contain a given vertex.

Question. *How are g and k related?*

We shall answer this question by computing the area of D in two different ways.

First approach. The areas of the surface Σ and the fundamental domain D are equal, and by the Gauss-Bonnet Theorem the area of the surface is given by $(-1)\,2\pi\,\chi(\Sigma) = 2\pi(2g-2) = 4g\pi - 4\pi$.

Second approach. The area of D is also given by its angular defect. Since D is a $4g-$gon and each vertex angle has measure $2\pi/k$, this angular defect is given by

$$(4g-2)\pi \;-\; 4g\left(\frac{2\pi}{k}\right) \;=\; 4g\pi \;-\; 2\pi \;-\; \frac{8g\pi}{k}\ .$$

If we equate these two expressions for the area, we see that

$$4g\pi \;-\; 2\pi \;-\; \frac{8g\pi}{k} \;=\; 4g\pi \;-\; 4\pi$$

which simplifies to

$$4g\pi \;=\; \frac{8g\pi}{k}\ .$$

If we solve this for k, we find that $k = 4g$.∎

Note that the formula $g = 4k$ also applies when $g = 1$ (see the comments about that case in the discussion of the tessellations).