Mathematics 246A, Winter 2009, Take-Home Examination

This will be due on

Wednesday, March 18, 2009, at 4:30 P.M.

either in my mailbox, at the front desk of Surge 202

or in an electronic format as indicated below.

You must show the work behind or reasons for your answers. The point values for problems are indicated in brackets.

Given a simplicial complex $\mathbf{K}$ and a subcomplex $\mathbf{L}$, the *relative Euler characteristic* $\chi(\mathbf{K},\mathbf{L})$ is defined to be

$$\sum_q (-1)^q \dim C_q(\mathbf{K},\mathbf{L})_{(0)}$$

in analogy with Corollary IV.3.11 and Theorem IV.3.12 of the notes (the latter only consider the case where $\mathbf{L}$ is empty). The cited results imply that this number is equal to

$$\sum_q (-1)^q \dim H_q(|\mathbf{K}|,|\mathbf{L}|)_{(0)}$$

where the objects on the right are rationalized singular homology groups and $|\mathbf{M}|$ denotes the underlying topological space of a simplicial complex $\mathbf{M}$.

1. [15 *points*] Explain why the relative Euler characteristic satisfies the following properties:

(a) If $|\mathbf{K}|$ and $|\mathbf{L}|$ are homotopy equivalent, then $\chi(\mathbf{K}) = \chi(\mathbf{L})$.

(b) If $\mathbf{L}$ is a subcomplex of $\mathbf{K}$, then $\chi(\mathbf{K}) = \chi(\mathbf{L}) + \chi(\mathbf{K},\mathbf{L})$.

(c) If $\mathbf{K}$ and $\mathbf{L}$ are subcomplexes of $\mathbf{M}$, then $\chi(\mathbf{K},\mathbf{K}\cap\mathbf{L}) = \chi(\mathbf{K}\cup\mathbf{L},\mathbf{L})$ or equivalently $\chi(\mathbf{K}\cup\mathbf{L}) = \chi(\mathbf{K}) + \chi(\mathbf{L}) - \chi(\mathbf{K}\cap\mathbf{L})$.

2. [30 *points*] Suppose that g is an integer valued function on simplicial complex pairs $(\mathbf{K},\mathbf{L})$ which satisfies the three properties in the preceding exercise. Prove that for all pairs $(\mathbf{K},\mathbf{L})$ we have $g(\mathbf{K},\mathbf{L}) = \chi(\mathbf{K},\mathbf{L}) \cdot g(\Delta_0)$. [*Hints:* In Property (c), what conclusion can we draw if the intersection is empty? Use this to dispose of the 0-dimensional case, first if $\mathbf{L}$ is empty and then more generally. Next, consider the following statements.

[A_k] The formula for g is true if $\mathbf{K}$ is at most k-dimensional (hence the same is true for $\mathbf{L}$).

[B_k] The formula for g is true for the pair $(\Delta_k, \partial\Delta_k)$.

[$C_{k,m}$] The formula for g is true if $\mathbf{K}$ is at most k-dimensional and the number of k-simplices in $\mathbf{K}$ but not in $\mathbf{L}$ is at most m.

Why is [A_k] equivalent to [$C_{k+1,0}$] when $\mathbf{L}$ is empty in the second case, and how can one extend this to the general case where $\mathbf{L}$ is not necessarily nonempty? Prove that [A_k] $\Longrightarrow$ [B_{k+1}] and [$C_{k,m}$] $\Longrightarrow$ [$C_{k,m+1}$] for all $m \geq 0$.]

3. [30 *points*] If $\mathbf{K}_1$ and $\mathbf{K}_2$ are simplicial complexes, then it is possible to construct a simplicial decomposition $\mathbf{K}_1 \prod \mathbf{K_2}$ on the space $|\mathbf{K}_1| \times |\mathbf{K}_2|$ such that if $\mathbf{L}$ is a subcomplex of $\mathbf{K}_1$, then $\mathbf{L} \prod \mathbf{K_2}$ is a subcomplex of $\mathbf{K}_1 \prod \mathbf{K_2}$. This construction is given in Section II.8 of Eilenberg and Steenrod; all we need to know about it for this exercise is stated in the previous sentence.

(a) Prove that the Euler characteristic satisfies

$$\chi\left(\mathbf{K}_1 \prod \mathbf{K_2}\right) \quad = \quad \chi(\mathbf{K}_1) \cdot \chi(\mathbf{K_2}) \; .$$

[*Hint:* Try to use the preceding exercise.]

(b) If $n \geq 2$ is an integer, prove that the spaces $S^n \times S^n$ and $S^{n-1} \times S^{n+1}$ are not homeomorphic (or even homotopy equivalent). Strictly speaking, there are two cases depending upon whether n is even or odd.

4. [25 *points*] Suppose that $(P, \mathbf{K})$ and $(Q, \mathbf{L})$ are polyhedra in $\mathbb{R}^n$ such that $\mathbf{p}$ is a vertex of $\mathbf{K}$ and $\mathbf{q}$ is a vertex of $\mathbf{L}$. Define the *wedge sum*

$$(P, \mathbf{p}) \vee (Q, \mathbf{q}) \quad = \quad (P \times \{\mathbf{q}\}) \, \cup \, (\{\mathbf{p}\} \times Q) \quad \subset \quad \mathbb{R}^{2n}$$

and take the simplicial decomposition of this space given by $\mathbf{K} \times \{\mathbf{q}\}$ and $\{\mathbf{p}\} \times \mathbf{L}$.

(a) Give a formula for the Euler characteristic of the wedge sum in terms of $\chi(P)$ and $\chi(Q)$.

(b) For each positive integer $n \geq 2$ construct a **connected** 2-dimensional polyhedron P_n whose Euler characteristic is equal to n. [*Note:* $\chi(\Delta_0) = 1$ and for each $m \leq 0$ there is a connected graph whose Euler characteristic is equal to m.]

Guidelines for electronic files

Any version of TEX that works easily on the Department's Unix machines (no rare or exotic, hard to find macros) is fine, including standard forms of LATEX. If you wish to send PostScript or `pdf` files, these are also options.

An extra credit problem will be posted later.