Addendum to Section VI.1

The following material should be inserted at the end of Section **VI.1** of the notes.

Simplicial cohomology

As before, let π be an abelian group.

Given a simplicial complex $(P, \mathbf{K})$ and a subcomplex $(Q, \mathbf{L})$, one can define the (unordered) simplicial cochain complex $C^*(\mathbf{K}, \mathbf{L}; \pi)$ to be $\mathrm{Hom}(C_*(\mathbf{K}, \mathbf{L}); \pi)$. These objects are contravariantly funtorial with respect to subcomplex inclusions, and as before one obtains long exact cohomology sequences for pairs. Furthermore, if we apply $\mathrm{Hom}(...; \pi)$ to the canonical natural maps $\lambda : C_*(\mathbf{K}, \mathbf{L}) \to S_*(P, Q)$, then we obtain canonical natural cochain complex maps

$$\psi : S^*(P, Q; \pi) \longrightarrow C^*(\mathbf{K}, \mathbf{L}; \pi)$$

and these in turn yield a commutative ladder diagram relating the long exact cohomology sequences for (P, Q) and $(\mathbf{K}, \mathbf{L})$. Previous experience suggests that the associated cohomology maps ψ^* should be isomorphisms, and we shall prove this below.

PROPOSITION 4A. *The maps ψ^* define isomorphisms relating the long exact cohomology sequences for (P, Q) and $(\mathbf{K}, \mathbf{L})$.*

Proof. Consider the functorial chain maps λ as above; we known these maps define isomorphisms in homology. By construction λ maps a free generator $\mathbf{v}_0 \cdots \mathbf{v}_q$ of $C_q(\mathbf{K}, \mathbf{L})$ to an affine singular q-simplex T for (P, Q); therefore, if $\mathbf{V}_*(\mathbf{K}, \mathbf{L})$ is the quotient of $S_*(P, Q)$ by the image of λ, then it follows that the chain group $\mathbf{V}_q(\mathbf{K}, \mathbf{L})$ is free abelian on a subset of free generators for $S_q(P, Q)$, and by the long exact homology sequence for the short exact sequence

$$0 \to C_* \to S_* \to \mathbf{V}_* \to 0$$

it follows that all homology groups of $\mathbf{V}_*(\mathbf{K}, \mathbf{L})$ are zero. We can now use Proposition VI.0 to conclude that $\mathbf{V}_*(\mathbf{K}, \mathbf{L})$ has a contracting chain homotopy D_*, and we can use the associated maps $\mathrm{Hom}(D_*, \pi)$ to conclude that for each π all the cohomology groups of the cochain complex $\mathrm{Hom}(\mathbf{V}_*, \pi)$ are also zero. If we now apply this observation to the long exact cohomology sequence associated to

$$0 \to \mathrm{Hom}(\mathbf{V}_*, \pi) \to \mathrm{Hom}(S_*, \pi) \to \mathrm{Hom}(C_*, \pi) \to 0$$

we see that the map $\psi : \mathrm{Hom}(S_*, \pi) \to \mathrm{Hom}(C_*, \pi)$ must also induce isomorphisms in cohomology.■

Given a simplicial complex $(P, \mathbf{K})$ and an ordering of its vertices, one can similarly define an ordered cochain complex $C^*_{\mathrm{ordered}}(P, \mathbf{K})$ and canonical cochain complex maps

$$\alpha : C^*(P, \mathbf{K}) \longrightarrow C^*_{\mathrm{ordered}}(P, \mathbf{K})$$

and an analog of the preceding argument then yields the following result:

COROLLARY 4B. *The associated maps in cohomology α^* are isomorphisms.*■

CUP PRODUCTS. If $\mathbb{D}$ is a commuative ring with unit, then one can define cup products on the cochain complexes $C^*(\mathbf{K}, \mathbb{D})$ using the same construction as in the singular case, and it is an elementary exercise to check that (a) this cup product has the previously described properties of the singular cup product, (b) the cochain map ψ preserves cup products at the cochain level (hence also in cohomology).

Examples of cochains

Formally speaking, cochains are fairly arbitrary objects, so we shall describe some "toy models" which reflect typical and important contexts in which concrete examples arise (also see Exercise VI.2 in `algtopexercises2010.pdf`). As usual, let $(P, \mathbf{K})$ be a polyhedron in $\mathbb{R}^n$, and let $f : P \to \mathbb{R}$ be a continuous function. We can then define a (simplicial) **line integral cochain** $\mathbf{L}_f \in C^1(\mathbf{K}; \mathbb{R})$ on free generators $\mathbf{v}_0\mathbf{v}_1$ by the formula

$$\mathbf{L}_f(\mathbf{v}_0\mathbf{v}_1) \quad = \quad \int_0^1 \; f\big(t\mathbf{v}_1 + (1-t)\mathbf{v}_0\big)\, |\mathbf{v}_1 - \mathbf{v}_0|\, dt \quad \in \quad \mathbb{R} \; .$$

By construction, this is just the scalar line integral of f along the directed straight line curve from $\mathbf{v}_0$ to $\mathbf{v}_1$.

Similarly, if $(P, \mathbf{K})$ is a polyhedron in $\mathbb{R}^3$ and $f : P \to \mathbb{R}$ is continuous, then we can define a surface integral cochain $\mathbf{S}_f \in C^2(\mathbf{K}; \mathbb{R})$ by the standard surface integral formula for scalar functions:

$$\mathbf{S}_f(\mathbf{v}_0\mathbf{v}_1\mathbf{v}_2) \quad = \quad \int_0^1 \int_0^{1-t} \; f(s\mathbf{v}_1 + t\mathbf{v}_2) \cdot |(\mathbf{v}_1 - \mathbf{v}_0) \times (\mathbf{v}_2 - \mathbf{v}_0)|\, ds\, dt$$

In this formula "$\times$" denotes the usual vector cross product. There are also versions of this construction in higher dimensions which yield cochains of higher dimension, but we shall not try to discuss them here.

Finally, given a field $\mathbb{F}$ we shall construct an explicit example of a cocycle in $C^1_{\mathrm{ordered}}(\partial\Delta_2; \mathbb{F})$ which is not a coboundary.

By construction $C_1^{\mathrm{ordered}}(\partial\Delta_2)$ is free abelian on free generators $\mathbf{e}_i\mathbf{e}_j$, where $0 \le i < j \le 2$. Thus a 1-dimensional cochain f is determined by its three values at $\mathbf{e}_0\mathbf{e}_1$, $\mathbf{e}_0\mathbf{e}_2$, and $\mathbf{e}_1\mathbf{e}_2$, each such cochain must be a cocycle because $C^2_{\mathrm{ordered}}(\partial\Delta_2; \mathbb{F})$ is trivial (hence $\delta^1 = 0$). Also, a cochain f is a coboundary if and only if there is some 0-dimensional cochain g such that

$$f(\mathbf{e}_i\mathbf{e}_j) \quad = \quad g(\mathbf{e}_i) \;-\; g(\mathbf{e}_j)$$

for all i and j such taht $0 \le i < j \le 2$.

Now consider the cochain f with $f(\mathbf{e}_0\mathbf{e}_1) = f(\mathbf{e}_0\mathbf{e}_2) = f(\mathbf{e}_1\mathbf{e}_2) = 1$. We claim that f cannot be a coboundary. If it were, then as above we could find integers $x_i = g(\mathbf{v}_i)$ such that

$$x_1 - x_0 \quad = \quad x_2 - x_0 \quad = \quad x_2 - x_1 \quad = \quad 1 \; .$$

This is a system of three linear equations in three unknowns, but it has no solutions. The nonexistence of solutions means that f cannot possibly be a coboundary. Similar considerations show that if k is an integer which is prime to the characteristic of $\mathbb{F}$ (in the characteristic zero case this means $k \neq 0$), then $k \cdot f$ is a cocycle which is not a coboundary.

By the previous results on cohomology isomorphisms, it follows that the singular cohomology $H^1(S^1; \mathbb{F})$ and simplicial cohomology $H^1(\partial\Delta_2; \mathbb{F})$ must also be nonzero.■