FIGURES FOR ALGEBRAIC TOPOLOGY LECTURE NOTES

I : Foundatonal and geometric background

I.2 : Barycentric coordinates and polyhedra

Barycentric coordinates. In the drawing below, each of the points **P, Q, R** lies in the plane determined by $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$, and consequently each can be written as a linear combination $w_1\mathbf{P}_1 + w_2\mathbf{P}_2 + w_3\mathbf{P}_3$, where $w_1 + w_2 + w_3 = 1$. In the picture below, for the point **P** the barycentric coordinates w_i are positive, while for the point **R** the barycentric coordinates are such that $w_1 = 0$ but the other two are positive, and for the point **Q** the barycentric coordinates are such that w_1 is negative but the other two are positive.

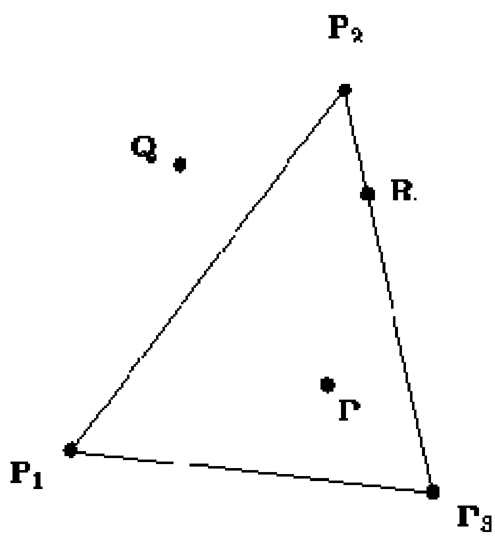


FIGURE 1

(***Source:*** http://graphics.idav.ucdavis.edu/education/GraphicsNotes/Barycentric-Coordinates/Barycentric-Coordinates.html)

Examples of points for which w_2 is positive but the remaining coordinates are negative can also be constructed using this picture; for example, if one draws the perpendicular from $\mathbf{P}_2$ to the line $\mathbf{P}_1\mathbf{P}_3$, then each point **S** on this perpendicular which lie above $\mathbf{P}_2$ in the picture (alternatively, $\mathbf{P}_2$ is between **S** and the foot of the perpendicular on $\mathbf{P}_1\mathbf{P}_3$) will have this property.

Illustration of a $\mathbf{2}$ ***– simplex .*** We shall use a modified version of Figure **1;** the points of the **2 –** simplex with vertices $\mathbf{P_1}$**,** $\mathbf{P_2}$**,** and $\mathbf{P_3}$ consists of the triangle determined by these points and the points which lie inside this triangle (in the usual intuitive sense of the word)**.**

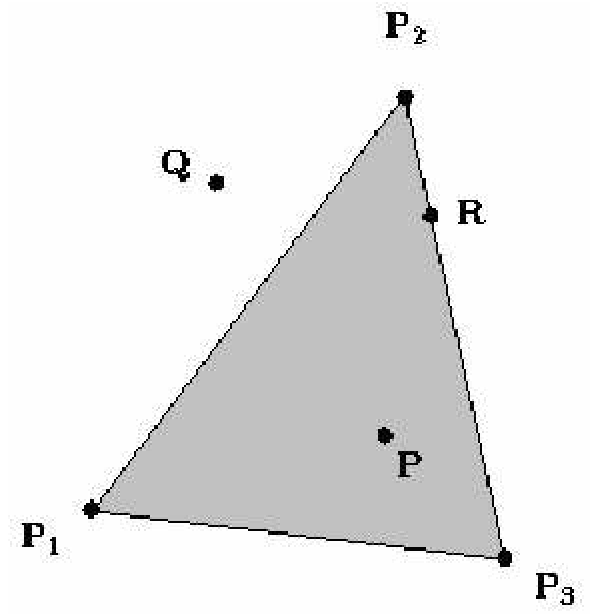
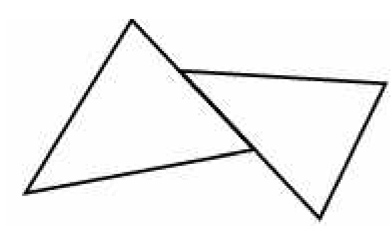


FIGURE 2

In this picture the points **P** and **R** lie on the simplex $\mathbf{P_1P_2P_3}$ because their barycentric coordinates are all nonnegative, but the point **Q** does not because one of its barycentric coordinates is negative**.**

Note that the **(*proper*)** ***faces*** of this simplex are the closed segments $\mathbf{P_1P_2}$**,** $\mathbf{P_2P_3}$**,** and $\mathbf{P_1P_3}$ joining pairs of vertices as well as the three vertices themselves (and possibly the empty set if we want to talk about an empty face with no vertices)**.**

Simplicial decompositions. It is useful to look at a few spaces given as unions of **2 –** simplices, some of which determine simplicial complexes in the sense of the notes and others that do not**.**



not a simplicial complex

FIGURE 3

(*Source:* http://mathworld.wolfram.com/SimplicialComplex.html**)**

In the example above the intersection of the $\mathbf{2}-$ simplices is not a common face. On the other hand, we can split the two simplices into smaller pieces such that we do have a simplicial decomposition**.**

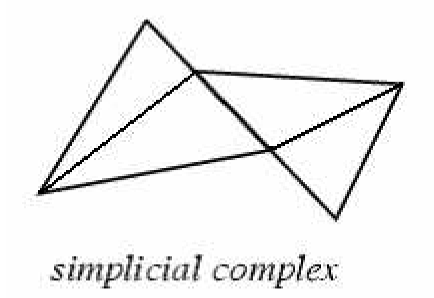
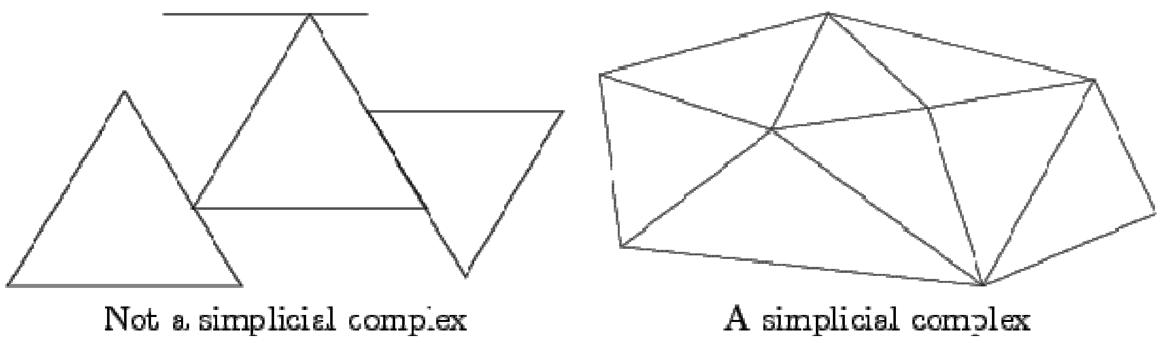


FIGURE 4

Here are two more examples; in the second case the simplices determine a simplicial complex and in the first they do not. As in the preceding example, one can subdivide the simplices in the first example to obtain a simplicial decomposition**.**



(*Source:* http://planning.cs.uiuc.edu/node274.html**)**

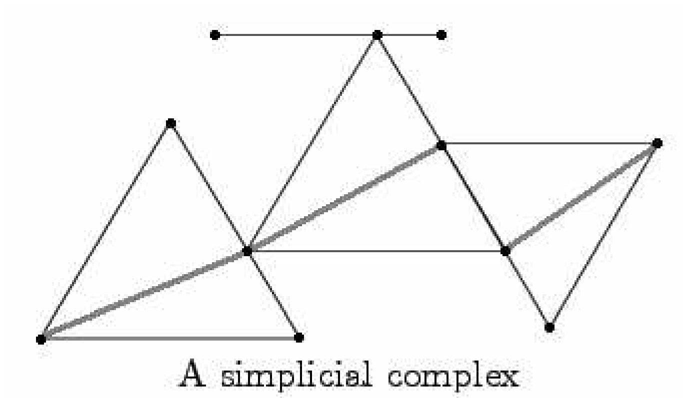


FIGURE 5

Triangulations. In the example from page 523 of Marsden and Tromba, the annulus bounded by two circles is split into four isometric pieces as in the drawing on the next page**.**

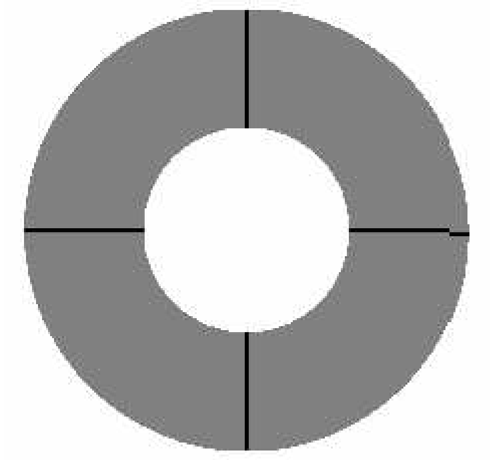


FIGURE 6

Each of the four pieces is homeomorphic to a solid rectangle. Since a solid rectangle has a simplicial decomposition into two $2-$ simplices, one can use such a decomposition to form a triangulation of the solid annulus.

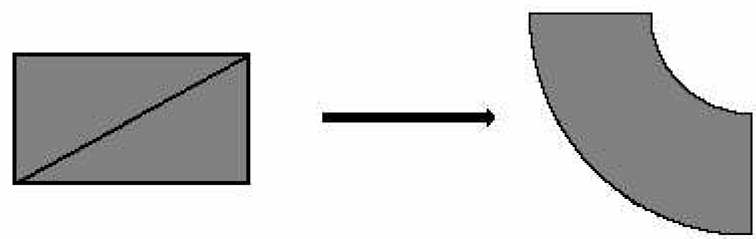


FIGURE 7

Another way of triangulating the annulus is suggested by the figure below:

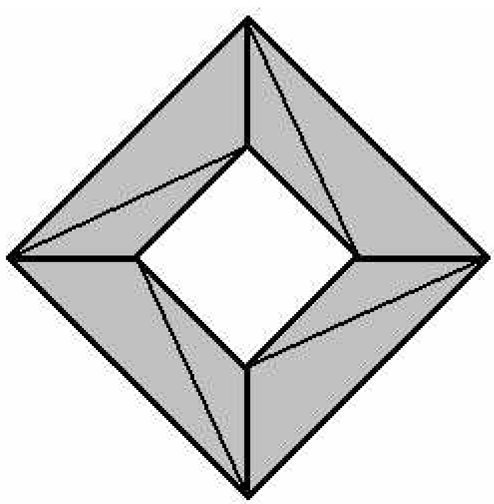


FIGURE 8

Similarly, many familiar closed polygonal regions can be triangulated fairly easily. Here is an example for a solid hexagon.

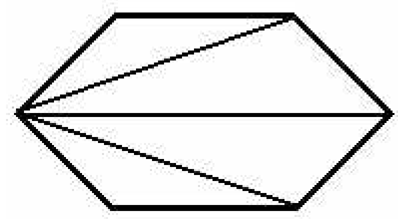


FIGURE 9

Triangulations of prisms. The drawings below illustrate the standard decomposition of a **3** – dimensional triangular prism**.**

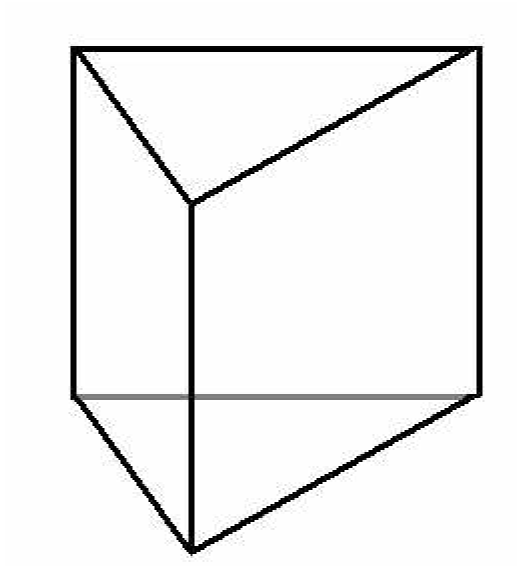


FIGURE 10

If we take $\mathbf{x_0}$**,** $\mathbf{x_1}$**,** and $\mathbf{x_2}$ to be the vertices of the bottom triangle and $\mathbf{y_0}$**,** $\mathbf{y_1}$**,** and $\mathbf{y_2}$ to be the vertices of the top triangle, then the decomposition is given as follows:

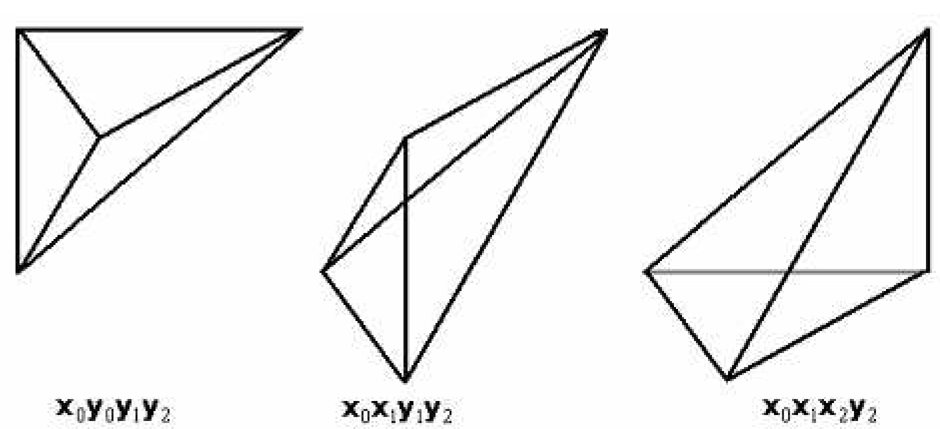


FIGURE 11

I.3 : Subdivisions

Simple subdivisions **1.** The drawing below depicts a subdivision of a $1-$ simplex given by a closed interval in the real line into three $1-$ simplices (which are just subintervals of the original interval).

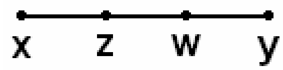


FIGURE 1

Similarly, every partition of an interval determines a subdivision.

Simple subdivisions **2.** The drawing below depicts a subdivision of a $2-$ simplex into two $2-$ simplices.

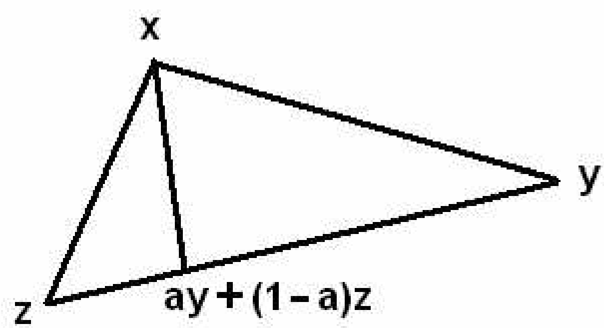


FIGURE 2

If $\mathbf{w} = a\mathbf{y} + (1-a)\mathbf{z}$ where $0 < a < 1$ and $p\mathbf{x} + q\mathbf{y} + r\mathbf{z}$ is a point on the simplex **xyz** (so that $p, q, r \geq 0$ and $p\mathbf{x} + q\mathbf{y} + r\mathbf{z} = 1$), then the point $p\mathbf{x} + q\mathbf{y} + r\mathbf{z}$ lies on the simplex **xwz** if and only if $p = 1$ or $p < 1$ and $q \geq a(1-p)$, and $p\mathbf{x} + q\mathbf{y} + r\mathbf{z}$ lies on the simplex **xwy** if and only if $p = 1$ or $p < 1$ and $q \leq a(1-p)$. The intersection of these simplices is the face with vertices **x** and **w.** Some other simple subdivisions of a $2-$ simplex are illustrated below.

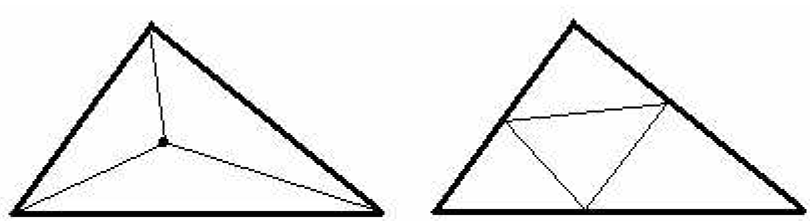


FIGURE 3

A nonexample. In general, if we given two simplicial decompositions, then neither is a subdivision of the other. For example, neither of the two simplicial decompositions of a rectangle described below is a subdivision of the other.

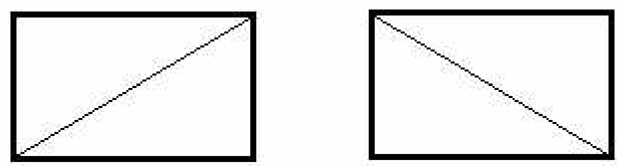


FIGURE 4

On the other hand, there is a decomposition which is a subdivision of ***both*** the decompositions shown above.

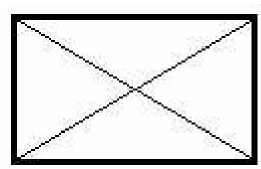


FIGURE 5

More generally, if we are given two simplicial decompositions $\mathbf{K}$ and $\mathbf{L}$ of a polyhedron **P** then one can always construct a third decomposition which is a subdivision of both $\mathbf{K}$ and $\mathbf{L}$. This follows from results in the book, ***Elementary Differential Topology,*** by J. R. Munkres (see the notes for a more complete citation).

Barycentric subdivisions. Here is a drawing to illustrate the barycentric subdivision of a $\mathbf{2}-$ simplex.

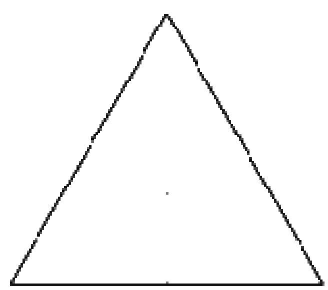


FIGURE 6

The vertices of a $\mathbf{2}-$ simplex in this subdivision are given by **a, b** and **c,** where **a** is a vertex of the original simplex, **b** is the midpoint of an edge which has **a** as a vertex, and **c** is the barycenter of the $\mathbf{2}-$ simplex itself. In this example, the diameters of the $\mathbf{2}-$ simplices in the barycentric subdivision are $\mathbf{2/3}$ the diameter of the original simplex.

The drawing below illustrates how the barycentric subdivision of a solid rectangular region which is decomposed into two $\mathbf{2}-$ simplices along a diagonal.

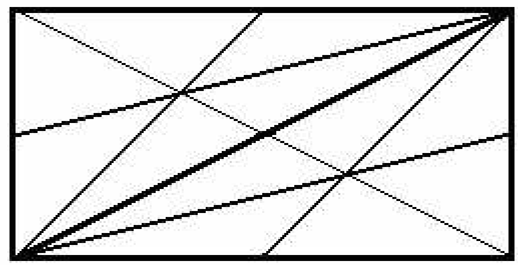


FIGURE 7

The next drawing illustrates the ***second barycentric subdivision*** of a $\mathbf{2}$ – simplex (however, the locations of several vertices are slightly inaccurate)**.**

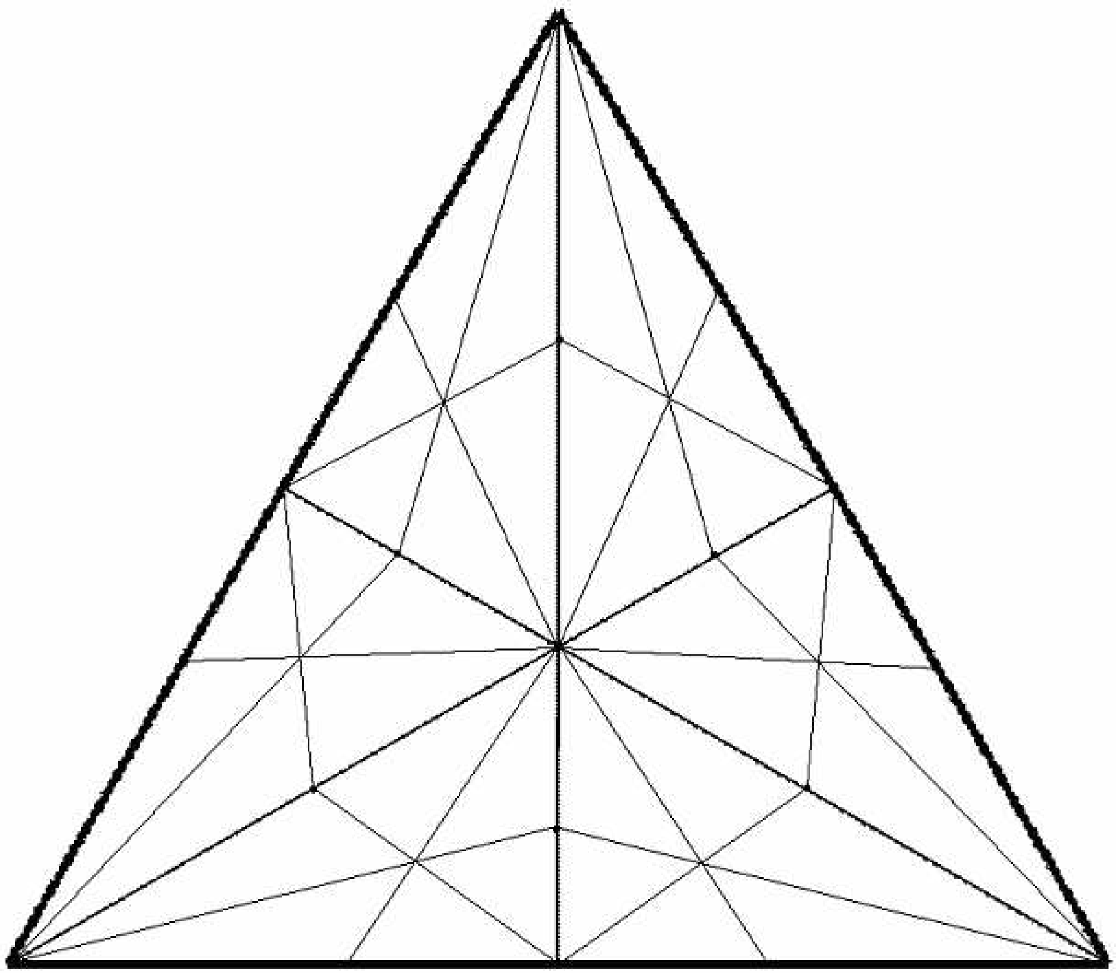


FIGURE 8

This decomposition has **25** vertices, **60** edges and **36** simplices that are **2** – dimensional**.** Incidentally, in the third barycentric subdivision there are **121** vertices, **336** edges and **216** simplices that are **2** – dimensional**.**