Math 246A
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Introduction to degree theory

If $n > 0$ and $f : S^n \to S^n$ is a continuous mapping, then the *degree* of f is the unique integer d such that the map $f_* : H_n(S^n) \to H_n(S^n)$ is multiplication by d (recall that $H_n(S^n) \cong \mathbf{Z}$ and every homomorphism of the latter to itself is multiplication by some integer).

Several properties of the degree are immediate:

(1) If f is the identity, then the degree of f is 1.

(2) If f is a constant map, then the degree of f is 0.

(3) If f and g are homotopic, then their degrees are equal.

(4) If f and g are continuous maps from S^n to itself, then the degree of $f \circ g$ is equal to the degree of f times the degree of g.

(5) If h is a homeomorphism of S^n to itself, then the degree of h and h^{-1} is ± 1, and the degree of $h \circ f \circ h^{-1}$ is equal to the degree of f.

(6) If $n = 1$ and $f(z) = z^m$ (complex arithmetic), then the degree of f is equal to m.

The last property is the only one which is nontrivial. It follows because (a) the map f_* from $\pi_1(S^1, 1) \cong \mathbf{Z}$ is multiplication by m, (b) the Hurewicz map from $\pi_1(S^1, 1)$ to $H_1(S^1)$ is an isomorphism, (c) the Hurewicz map defines a natural transformation of functors from the fundamental group to 1-dimensional singular homology.■

For all $n \geq 2$, there is a standard recursive process for constructing continuous maps from S^n to itself with arbitrary degree.

PROPOSITION 1. *Let $f : S^{n-1} \to S^{n-1}$ be a continuous mapping of degree d, and let its* suspension *$\Sigma(f) : S^n \to S^n$ be defined on $(x, t) \in S^n \subset \mathbf{R}^n \times \mathbf{R}$ by*

$$\Sigma(f)(x,t) \quad = \quad \left(\sqrt{1-t^2} \cdot f(x),\ t \right) \ .$$

Then $\Sigma(f)$ is continuous and its degree is also equal to d.

COROLLARY 2. *If $n \geq 1$ and d is an arbitrary integer, then there exists a continuous mapping $g : S^n \to S^n$ whose degree is equal to d.*

The case $n = 1$ of the corollary is just (6), above, and the proposition supplies the inductive step to show that if the corollary is true for $(n-1)$ then it is also true for n.■

Proof of Proposition 1. We should check first that the map $\Sigma(f)$ is continuous. This is immediate from the formula for all points except the north and south poles, and at the latter one can check directly that if $\varepsilon > 0$ then we can take $\delta = \varepsilon$.

Define D^n_+ and D^n_- to be the subsets of S^n on which the last coordinates are nonnegative and nonpositive respectively. It follows immediately that S^n is formed from S^{n-1} by attaching two n-cells corresponding to $D^n_\pm$. This and the vanishing of the homology of disks in positive dimensions imply that all the arrows in the diagram below are isomorphisms:

$$H_{*-1}(S^{n-1}) \to H_*(D^n_+, S^{n-1}) \leftarrow H_*(S^n, D^n_-) \to H_*(S^n)$$

Furthermore, the mappings f and $\Sigma(f)$ determine homomorphisms from each of these homology groups to themselves such that the following diagram commutes:

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It follows immediately that the degrees of f and $\Sigma(f)$ must be equal.∎

Here is another basic property:

PROPOSITION 3. *If $f : S^n \to S^n$ is continuous and the degree of f is nonzero, then f is onto.*

Proof. If the image of f does not include some point $\mathbf{p}$, then f_* has a factorization of the form

$$H_n(S^n) \ \to\ H_n(S^n - \{\mathbf{p}\}) \ \to\ H_n(S^n)$$

and this homomorphism is trivial because the middle group is zero.∎

Linear algebra and degree theory

We shall start with orthogonal transformations.

PROPOSITION 4. *Suppose that T is an orthogonal linear transformation of $\mathbf{R}^n$, where $n \geq 2$, and let $f_T : S^{n-1} \to S^{n-1}$ be the corresponding homeomorphism of S^{n-1}. Then the degree of f_T is equal to the determinant of T.*

Sketch of proof. We shall use a basic fact about orthogonal matrices; namely, if A is an orthogonal matrix then there is another orthogonal matrix B such that $B \cdot A \cdot B^{-1}$ is equal to a block sum of 2×2 rotation matrices plus a block sum of 1×1 matrices such that at most one of the latter has an entry of -1 (and the rest must have entries of 1).

Every 2×2 rotation matrix can be joined to the identity by a path consisting entirely of 2×2 rotation matrices. Therefore it follows that f_T is homotopic to f_S, where S is a diagonal matrix with at most one entry equal to -1 and all others equal to 1. Clearly the degrees of f_S and f_T are equal, and likewise the determinants of S and T must be equal (by continuity of the determinant and the fact that its value for an orthogonal matrix is always ± 1). Thus the proof reduces to showing that the degree of f_S is equal to -1 if there is a negative diagonal entry and is equal to 1 if there are no negative diagonal entries. — In fact, the second statement is obvious since T and f_T are identity mappings in this case.

Therefore everything reduces to showing that the degree of f_S is equal to -1. We can use Proposition 2 to show that the result is true for all n if it is true for $n = 2$, and the truth of the result when $n = 2$ follows immediately from Property (6) of degrees that was stated at the beginning of this document.∎

We shall now consider an arbitrary invertible linear transformation T from $\mathbf{R}^n$ to itself. Such a map is a homeomorphism and thus extends to a map $T^\bullet$ of one point compactifications from S^n to itself.

THEOREM 5. *In the setting above, the degree of $T^\bullet$ is equal to the sign of the determinant of T.*

The proof of this result requires some additional input.

LEMMA 6. *Suppose that we are given a continuous curve T_t defined for $t \in [0,1]$ and taking values in the set of all invertible linear transformations on $\mathbf{R}^n$ (equivalently, invertible $n \times n$ matrices). Then $T_0^\bullet$ is homotopic to $T_1^\bullet$.*

Proof of Lemma 6. We would like to define a homotopy by the formula $H_t = T_t^\bullet$, and we can do so if and only if the latter is continuous at every point of $\{\infty\} \times [0,1]$. The latter in turn reduces to showing the following: *For each $t \in [0,1]$ and $M > 0$ there are numbers $\delta > 0$ and $P > 0$ such that $|s - t| < \delta$ and $|v| \geq P$ imply $|T_s(v)| \geq M$.*

Let $\|T\|$ be the usual norm of a linear transformation given by the maximum value of $|T|$ on the unit sphere. It follows immediately that the norm is a continuous function in (the matrix entries associated to) T. It follows that

$$|T_s(v)| \quad \geq \quad \|T_s^{-1}\| \cdot |v|$$

and since the inverse operation is also continuous it follows that $\|T_s^{-1}\|$ is a continuous function of s. In particular, if $\|T_t^{-1}\| = B > 0$ then we can find $\delta > 0$ such that $|s - t| < \delta$ implies $\|T_s^{-1}\| > B/2$, and hence if $|v| > 2M/B$ and $|t - s| < \delta$ then $T_s(v)| \geq M$, as required.■

Proof of Theorem 5. Both the degree of $T^\bullet$ and the sign of the determinant are homomorphisms from invertible matrices to $\{\pm 1\}$, and therefore it will suffice to prove the theorem for a set of linear transformations which generate all the invertible linear transformations. Not surprisingly, we shall take this set to be the linear transformations given by the elementary matrices.

Let $E_{i,j}$ denote the $n \times n$ matrix which has a 1 in the (i,j) entry and zeros elsewhere. Then the function sending $t \in [0,1]$ to $I + tE_{i,j}$ defines a curve from the elementary matrix $I + E_{i,j}$ to the identity. Therefore the associated linear transformation determines a map which is homotopic to the identity, and consequently the degree and determinant sign agree for elementary linear transformations given by adding a multiple of one row to another.

Similarly, if $D(k,r)$ is a diagonal matrix which has ones except in the k^{th} position and a positive real number r in the latter position, then there is a continuous straight line curve joining the matrix in question to the identity, and this matrix takes values in the group of invertible diagonal matrices. It follows that the degree and determinant sign agree for elementary linear transformations given by multiplying one row by a positive constant.

We are now left with elementary matrices given by either multiplying one row by -1 or by interchanging two rows. These two types of matrices are similar, so both the degrees and determinant signs are equal in each case. Therefore it will suffice to check that the degree and determinant sign agree when one considers an elementary matrix given by multiplying a single row by -1.

By Proposition 2 and the invariance of our numerical invariants under similarity, it will suffice to consider the case where $n = 2$ and we are multiplying the second row by -1. Let $W \subset \mathbf{R}^2$ be the open disk of radius 2 about the origin, so that there is a canonical homeomorphism from $W - \{\mathbf{0}\}$ to $S^1 \times (0,2)$. Now the map $T^\bullet$ sends $S^2 - \{\mathbf{0}\}$ to itself and likewise for W and S^1. Excision and homotopy invariance now yield the following chain of isomorphic homology groups:

$$H_1(S^1) \;\leftarrow\; H_1(W - \{\mathbf{0}\}) \;\rightarrow\; H_2(W, W - \{\mathbf{0}\}) \;\leftarrow\; H_2(S^2, S^2 - \{\mathbf{0}\}) \;\longrightarrow\; H_2(S^2)$$

As in Proposition 3, one has associated maps of homology groups to form a corresponding commutative diagram, and from this diagram one sees that the degree of $T^\bullet$ is equal to the degree of the map determined by $T^\bullet$ on S^1. Since the map on S^1 is merely the mapping sending z to z^{-1}, it follows that the degree is equal to -1, and of course this is the same as the sign of the determinant.■

The Fundamental Theorem of Algebra

One can use degree theory to prove the Fundamental Theorem of Algebra. All proofs of the latter involve some analysis and plane topology, and one advantage of the degree-theoretic proof is that the role of topology is particularly easy to recognize. This proof can also be modified to obtain a generalization of the Fundamental Theorem of Algebra to polynomials with quaternionic coefficients (this was done by Eilenberg and Niven in the 1940s).

We start with an argument that is similar to the proof in the last part of Theorem 5.

PROPOSITION 7. *The map ψ^m of the complex plane sending z to z^m (where m is a positive integer) extends continuously to a map of one point compactifications sending the point at infinity to itself, and the degree of the compactified map is equal to m.*

Proof. The existence of a continuous extension follows because if $M > 0$ then $|z| > M^{1/m}$ implies $|z^m| > M$.

It follows that ψ^m sends $\mathbf{C} - \{\mathbf{0}\}$ to itself. Of course, the map also sends S^1 to itself and this map has degree m, so a diagram chase plus the naturality of the Hurewicz homomorphism imply that ψ^m_* is multiplication by m on $H_1(\mathbf{C} - \{\mathbf{0}\}) \cong \mathbf{Z}$. Diagram chases now show that ψ_* is multiplication by m on

$$H_2(\mathbf{C}, \mathbf{C} - \{\mathbf{0}\}) \quad \cong \quad H_2(S^2, S^2 - \{\mathbf{0}\}) \quad \cong \quad H_2(S^2)$$

and thus the degree of the compactified map is equal to m.■

The following result is standard.

PROPOSITION 8. *If p is a nonconstant monic polynomial, then p extends continuously to a map of one point compactifications sending the point at infinity to itself.*

Sketch of proof. We need to show that if $M > 0$ then there is some $\rho > 0$ such that $|z| > \rho$ implies $|p(z)| > M$. One easy way of doing this is to begin by writing p as follows:

$$p(z) \quad = \quad z^m \cdot \left(1 \; + \; \frac{a_{n-1}}{z} \; + \; \cdots \; + \; \frac{a_0}{z^n}\right)$$

If we write the expression inside the parentheses as $1 + b(z)$, then it is clear that if $|z|$ is sufficiently large (say $|z| > N$) then $|b(z)| < \frac{1}{2}$. It follows immediately that if $M > 0$ and $|z| > 2M^{1/m} + N$ then $|p(z)| > M$.■

The Fundamental Theorem of Algebra will now be a consequence of Proposition 3 and the following generalization of Proposition 8:

PROPOSITION 9. *If p is a nonconstant monic polynomial of degree $m \geq 1$, then the degree of the compactified map $p^\bullet$ is equal to m.*

Proof. It will suffice to show that $p^\bullet$ is homotopic to $(\psi^m)^\bullet$.

Define a homotopy from ψ^m to p on the set where $|z| \geq N + 1$ by $h_t(z) = z^m(1 + t\, b(z)\,)$. By the Tietze Extension Theorem, one can extend this to a homotopy over all of $\mathbf{C}$. As in the previous argument, if $M > 0$ and $|z| > 2M^{1/m} + N + 1$ then $|h_t(z)| > M$ for all t. One can then argue as in the first paragraph of the proof of Lemma 6 to show that $p^\bullet$ is homotopic to $(\psi^m)^\bullet$.■