Math 246A
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Comments on the Jordan-Brouwer Separation Theorem

This is a written version which includes some information not covered in Hatcher.

JORDAN-BROUWER SEPARATION THEOREM. *Let $n \geq 2$, and suppose that $A \subset S^n$ is homeomorphic to S^{n-1}. Then $S^n - A$ contains two components, and A is the frontier of each component.*

Note on the proof. The existence of two components is shown in Hatcher. We only need to prove that points of A are limit points of each components. Suppose that $S^n - A$ is the union of the two open, connected, disjoint subsets U and V.

Assume that not every point of A is a limit point of both U and V. Without loss of generality, it is enough to consider the case where $x \in A$ is not a limit point of V. Since $x \notin V$, it follows that there is some open set W_0 in S^n such that $x \in W_0$ and $W_0 \cap V = \emptyset$.

Consider the open set $W_0 \cap A$ in A; since the latter is homeomorphic to S^{n-1}, it follows that there is a subneighborhood of the form $A - E$, where $E \subset A$ is homeomorphic to a closed $(n-1)$-disk and $A - E$ is homeomorphic to an open $(n-1)$-disk centered at x. If $W = W_0 \cap S^n - E$, then W is still open in S^n and we still have $x \in W$ and $W \cap V = \emptyset$.

By construction we have $S^n - E = U \cup A - E \cup V$ where the pieces are pairwise disjoint. Furthermore, we have $A - E \subset W$ and hence $U \cup W$ is an open set of $S^n - E$ which is disjoint from V and contains U and $A - E$. Therefore it follows that $S^n - E$ is a union of the nonempty disjoint open sets $U \cup W$ and V and hence is disconnected. On the other hand, since E is homeomorphic to a closed disk we know that $S^n - E$ is connected, so we have a contradiction. The source of this contradiction was our assumption that x was not a limit point of V, and hence this must be false. Therefore x must be a limit point of V, and as noted above it follows that every point of A is a limit point of both U and V.■