Mathematics 246A, Winter 2007, Take-Home Examination

This will be due on

Wednesday, March 21, 2007, at 2:30 P.M.

either in my mailbox or at the front desk of Surge 202.

You must show the work behind or reasons for your answers. Each problem is worth 20 *points.*

1. (i) Let X be a nonempty topological space, and let $n \geq 0$. Prove by induction on n that $H_q(S^n \times X) \cong H_q(X) \oplus H_{q-n}(X)$ for all integers q. [*Hint:* Look at the Mayer-Vietoris sequence arising from $S^n \times X = U \times X \cup V \times X$, where $U = S^n - \{\mathbf{e}\}$ and $V = S^n - \{-\mathbf{e}\}$ for some unit vector $\mathbf{e}$. What do we know about the homotopy types of U, V and $U \cap V$, and how can we exploit this?]

(ii) Let n_1 and n_2 be positive. Explain why $S^{n_1} \times S^{n_2}$ is not homeomorphic to $S^{n_1+n_2}$.

2. (i) Suppose that we are given simplicial complexes $(P, \mathbf{K})$ and $(Q, \mathbf{L})$ in $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and suppose that $\mathbf{x}$ and $\mathbf{y}$ are vertices of $\mathbf{K}$ and $\mathbf{L}$ respectively. Define the *wedge*

$$(P, \mathbf{K}; \mathbf{x}) \ \vee \ (Q, \mathbf{L}; \mathbf{y})$$

to be the simplicial complex whose underlying space is

$$P \times \{\mathbf{y}\} \ \cup \ \{\mathbf{x}\} \times Q \quad \subset \quad \mathbf{R}^n \times \mathbf{R}^m \quad \cong \quad \mathbf{R}^{n+m}$$

and whose simplices have one of the forms $A \times \{\mathbf{y}\}$ or $\{\mathbf{x}\} \times Q$, where A is a simplex of $\mathbf{K}$ or B is a simplex of $\mathbf{L}$. Find a formula for the Euler characteristic of the wedge in terms of $\chi(P)$ and $\chi(Q)$. [*Hint:* How many simplices are there in a fixed dimension k? There are two cases depending upon whether or not k is positive. Check your formula using some examples.]

(ii) Given an arbitrary integer p, prove that there is a finite connected simplicial complex whose Euler characteristic is equal to p. [*Hint:* There are obvious examples when p is 0, 1 or 2. How can one combine this with the result in the first part?]