Math 246A
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Comments on the Brouwer Fixed Point Theorem

The purpose of this document is twofold. First, we shall give a complete and rigorous proof of a key step in the proof of the Brouwer Fixed Point Theorem that is often treated only in an intuitive manner (*e.g.*, on page 32 of Hatcher). Second, we shall include a proof of the result on eigenvalues of matrices with positive entries that was covered in class.

The retraction in Brouwer's Theorem

As indicated on page 32 of Hatcher, the idea is simple. We start with two distinct points $\mathbf{x}$ and $\mathbf{y}$ on the disk D^n and consider the ray starting with $\mathbf{y}$ and passing through $\mathbf{x}$; algebraically, this is the set of all points expressible as $\mathbf{y} + (1-t)\mathbf{x}$, where $t \geq 0$. Simple pictures strongly suggest that there is a unique scalar $t \geq 1$ such that $\mathbf{y} + (1-t)\mathbf{x}$ lies on S^{n-1}, if $\mathbf{x} \in S^{n-1}$ then $t = 1$ so that the point is equal to $\mathbf{x}$, and in fact the value of t is a continuous function of $(\mathbf{x}, \mathbf{y})$. Our purpose here is to justify these assertions.

PROPOSITION. *There is a continuous function* $\rho : D^n \times D^n - \text{Diagonal} \to S^{n-1}$ *such that* $\rho(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ *if* $\mathbf{x} \in S^{n-1}$.

If we have the mapping ρ and f is a continuous map from D^n to itself without fixed points, then the retraction from D^n onto S^{n-1} is given by $\rho(\mathbf{x}, f(\mathbf{x}))$.

Proof of the proposition. It follows immediately that the intersection points of the line joining $\mathbf{y}$ to $\mathbf{x}$ are give by the values of t which are roots of the equation

$$|\mathbf{y} \ + \ t(\mathbf{x} - \mathbf{y})|^2 \ = \ 1$$

and the desired points on the ray are given by the roots for which $t > 1$. We need to show that there is always a unique root satisfying this condition, and that this root depends continuously on $\mathbf{x}$ andn $\mathbf{y}$.

We can rewrite the displayed equation as

$$|\mathbf{x} - \mathbf{y}|^2 t^2 \ + \ 2\langle \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle t \ + \ (|\mathbf{y}|^2 - 1) \ = \ 0 \ .$$

If try to solve this nontrivial quadratic equation for t using the quadratic formula, then we obtain the following:

$$t \ = \ \frac{-\langle \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle \ \pm \ \sqrt{\langle \mathbf{y}, \mathbf{x} - \mathbf{y} \rangle^2 \ + \ |\mathbf{x} - \mathbf{y}|^2 \cdot (1 - |\mathbf{y}|^2)}}{|\mathbf{x} - \mathbf{y}|^2}$$

One could try to analyze these roots by brute force, but it will be more pleasant to take a more qualitative viewpoint.

(*a*) *There are always two distinct real roots.* We need to show that the expression inside the square root sign is always a positive real number. Since $|\mathbf{y}| \leq 1$, the expression is clearly nonnegative, so we need only eliminate the possibility that it might be zero. If this happens, then each summand must be zero, and since $|\mathbf{y}-\mathbf{x}| > 0$ it follows that we must have both $\langle \mathbf{y}, \mathbf{x}-\mathbf{y}\rangle = 0$ and $1 - |\mathbf{y}|^2 = 0$. The second of these implies $|\mathbf{y} = 1$, and the first then implies

$$\langle \mathbf{y}, \mathbf{x}\rangle \quad = \quad |\mathbf{y}|^2 \quad = \quad 1 \; .$$

If we combine this with the Cauchy-Schwarz Inequality and the basic condition $|\mathbf{x}| \leq 1$, we see that $|\mathbf{x}|$ must equal 1 and $\mathbf{x}$ must be a positive multiple of $\mathbf{y}$; these in turn imply that $\mathbf{x} = \mathbf{y}$, which contradicts our hypothesis that $\mathbf{x} \neq \mathbf{y}$. Thus the expression inside the radical sign is positive and hence there are two distinct real roots.

(*b*) *There are no roots t such that $0 < t < 1$.* The Triangle Inequality implies that

$$|\mathbf{y} \; + \; t(\mathbf{x}-\mathbf{y})| \quad = \quad |(1-t)\mathbf{y} \; + \; t\mathbf{x}| \quad \leq \quad (1-t)|\mathbf{y}| \; + \; t|\mathbf{x}| \quad \leq 1$$

so the value of the quadratic function

$$q(t) \quad = \quad |\mathbf{x}-\mathbf{y}|^2 t^2 \; + \; 2\langle \mathbf{y}, \mathbf{x}-\mathbf{y}\rangle t \; + \; (|\mathbf{y}|^2 - 1)$$

lies in $[-1, 0]$ if $0 < t < 1$. Suppose that the value is zero for some t_0 of this type. Since there are two distinct roots for the associated quadratic polynomial, it follows that the latter does not take a maximum value at t_0, and hence there is some t_1 such that $0 < t_1 < 1$ and the value of the function at t_1 is positive. This contradicts our observation about the behavior of the function, and therefore our hypothesis about the existence of a root like t_0 must be false.

(*d*) *There is one root of $q(t)$ such that $t \leq 0$ and a second root such that $t \geq 1$.* We know that $q(0) \leq 0$ and that the limit of $q(t)$ as $t \to -\infty$ is equal to $+\infty$. By continuity there must be some $t_1 \leq 0$ such that $q(t_1) = 0$. Similarly, we know that $q(1) \leq 0$ and that the limit of $q(t)$ as $t \to +\infty$ is equal to $+\infty$, so again by continuity there must be some $t_2 \geq 1$ such that $q(t_2) = 0$.

(*d*) *The unique root t satisfying $t \geq 1$ is a continuous function of $\mathbf{x}$ and $\mathbf{y}$.* This is true because the desired root is given by taking the positive sign in the expression obtained from the quadratic formula, and it is a routine algebraic exercise to check that this expression is a continuous function of $(\mathbf{x}, \mathbf{y})$.

(*e*) *If $|\mathbf{x}| = 1$, then $t = 1$.* This just follows because $|\mathbf{y} \; + \; 1(\mathbf{x}-\mathbf{y})| = 1$ in this case.

The proposition now follows by taking

$$\rho(\mathbf{x}, \mathbf{y}) \quad = \quad \mathbf{y} \; + \; t(\mathbf{x}-\mathbf{y})$$

where t is given as above by taking the positive sign in the quadratic formula. The final property shows that $\rho(\mathbf{x}, \mathbf{y}) = \mathbf{x}$ if $|\mathbf{x}| = 1$.■

An eigenvector theorem for matrices with nonnegative entries

The first step is the following elementary fact:

LEMMA. *Let X be a topological space which is homeomorphic to D^n for some $n \geq 0$. Then every continuous map $f : X \to X$ has a fixed point.*

Proof. Let $f : X \to X$ be continuous, and let $h : X \to D^n$ be a homeomorphism. Then $h \circ f \circ h^{-1}$ is a continuous map from D^n to itself and thus has a fixed point $\mathbf{p}$ by Brouwer's Theorem. In other words we have $h \circ f \circ h^{-1}(\mathbf{p}) = \mathbf{p}$. If we take $\mathbf{q} = h(\mathbf{p})$, straightforward computation shows that $f(\mathbf{q}) = \mathbf{q}$.■

THEOREM. *Let $n > 1$, and let A be an $n \times n$ matrix which is invertible and has nonnegative entries. Then A has a positive eigenvalue λ such that λ has a nonzero eigenvector with nonnegative entries.*

Proof. Recall that the 1-norm on $\mathbf{R}^n$ is defined by $|\mathbf{x}|_1 = \sum_j |x_j|$, where the coordinates of $\mathbf{x}$ are given by $X_1, \cdots, x_n$. For each $\mathbf{x} \in \Delta_{n-1}$ define

$$f(\mathbf{x}) \quad = \quad (|A\mathbf{x}|_1)^{-1} \cdot A\mathbf{x} \ .$$

Observe that the coordinates of $A\mathbf{x}$ are all nonnegative because the entries of A and the coordinates of $\mathbf{x}$ are nonnegative, this vector is nonzero because A is invertible, and if $\mathbf{y}$ is a nonzero vector with nonnegative entries then $|\mathbf{y}|_1^{-1}\mathbf{y}$ must lie in Δ_{n-1}. Therefore we indeed have a continuous map f from the simplex to itself.

By the lemma, we know that f has a fixed point; in other words, there is some $\mathbf{v} \in \Delta_{n-1}$ such that

$$\mathbf{v} \quad = \quad (|A\mathbf{v}|_1)^{-1} \cdot A\mathbf{v}$$

and since the latter is equivalent to saying that $A\mathbf{v}$ is a positive multiple of $\mathbf{v}$, this completes the proof.■