Math 246A
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**Comments on the Jordan-Brouwer Separation Theorem**

This is a written version which includes some information not covered in Hatcher.

**JORDAN-BROUWER SEPARATION THEOREM.** *Let $n \geq 2$, and suppose that $A \subset S^n$ is homeomorphic to $S^{n-1}$. Then $S^n - A$ contains two components, and $A$ is the frontier of each component.*

**Note on the proof.** The existence of two components is shown in Hatcher. We only need to prove that points of $A$ are limit points of each components. Suppose that $S^n - A$ is the union of the two open, connected, disjoint subsets $U$ and $V$.

Assume that not every point of $A$ is a limit point of both $U$ and $V$. Without loss of generality, it is enough to consider the case where $x \in A$ is not a limit point of $V$. Since $x \notin V$, it follows that there is some open set $W_0$ in $S^n$ such that $x \in W_0$ and $W_0 \cap V = \emptyset$.

Consider the open set $W_0 \cap A$ in $A$; since the latter is homeomorphic to $S^{n-1}$, it follows that there is a subneighborhood of the form $A - E$, where $E \subset A$ is homeomorphic to a closed $(n-1)$-disk and $A - E$ is homeomorphic to an open $(n-1)$-disk centered at $x$. If $W = W_0 \cap S^n - E$, then $W$ is still open in $S^n$ and we still have $x \in W$ and $W \cap V = \emptyset$.

By construction we have $S^n - E = U \cup A - E \cup V$ where the pieces are pairwise disjoint. Furthermore, we have $A - E \subset W$ and hence $U \cup W$ is an open set of $S^n - E$ which is disjoint from $V$ and contains $U$ and $A - E$. Therefore it follows that $S^n - E$ is a union of the nonempty disjoint open sets $U \cup W$ and $V$ and hence is disconnected. On the other hand, since $E$ is homeomorphic to a closed disk we know that $S^n - E$ is connected, so we have a contradiction. The source of this contradiction was our assumption that $x$ was not a limit point of $V$, and hence this must be false. Therefore $x$ must be a limit point of $V$, and as noted above it follows that every point of $A$ is a limit point of both $U$ and $V$.■