The Steiner surface

This is a solution to the following exercise in Conlon:

EXERCISE. *Let $p : S^2 \to \mathbf{RP}^2$ be the usual double covering map, and let $g : \mathbf{RP}^2 \to \mathbf{R}^2$ be the unique smooth map such that g maps the equivalence class of $(x,y,z) \in S^2$ to $xy, yz, xz)$. Prove that g is an immersion on* $\mathbf{RP}^2 -$ (six points).

SOLUTION.

Cover $\mathbf{RP}^2$ with the coordinate charts h_1, h_2 and h_3 from the open unit disk

$$\{ (u,v) \mid u^2+v^2 < 1 \}$$

to $\mathbf{RP}^2$ such that

$$\begin{aligned} h_1(u,v) &= p\big(u, v, \sqrt{1-(u^2+v^2)}\ \big), \\ h_2(u,v) &= p(u, , \sqrt{1-(u^2+v^2)}, v), \\ h_3(u,v) &= p(\sqrt{1-(u^2+v^2)}, u, v). \end{aligned}$$

In principle, it suffices to find the points $h_i(u,v)$ such that g is not an immersion at $h_i(u,v)$ for $1 \leq i \leq 3$, and for each i this amounts to finding the points (u,v) such that $g \circ h_i$ is not an immersion at (u,v). We need to show that there are six points in $\mathbf{RP}^2$ which can be so described.

We shall concentrate on the case $i = 1$, and later we shall explain how one can use similar considerations apply to the other cases. Therefore set $\sigma = g \circ h_1$. It follows that the set of all points in the image of h_1 such that g is not a smooth immersion will be equal to the set of all points $h_1(u,v)$ such that σ is not an immersion at (u,v), which amounts to solving the vector equation

$$\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \quad = \quad \mathbf{0} \ .$$

It will simplify our equations if we set $w = \sqrt{1-(u^2+v^2)}$. **Note that w is positive, and in particular it is nonzero.**

Direct computation now yields

$$\frac{\partial \sigma}{\partial u} \quad = \quad \begin{pmatrix} v \\ \frac{-vu}{w} \\ \frac{w^2-u^2}{w} \end{pmatrix} \qquad\qquad \frac{\partial \sigma}{\partial v} \quad = \quad \begin{pmatrix} u \\ \frac{w^2-v^2}{w} \\ \frac{-uv}{w} \end{pmatrix} \ .$$

Therefore the cross product is given by

$$\frac{\partial \sigma}{\partial u} \times \frac{\partial \sigma}{\partial v} \quad = \quad \begin{pmatrix} \frac{u^2v^2-(w^2-u^2)(w^2-v^2)}{w^2} \\ \frac{u(w^2-u^2)+uv^2}{w} \\ \frac{v(w^2-v^2)+vu^2}{w} \end{pmatrix} \ .$$

We need to determine when all three coordinates of the vector on the right hand side are equal to zero. The first coordinate is equal to zero if and only if

$$u^2v^2 \quad = \quad (w^2 - u^2)(w^2 - v^2)$$

and this reduces to

$$0 \quad + \quad w^4 - w^2u^2 - w^2v^2 \quad = \quad w^2(w^2 - u^2 - v^2)$$

so that $w > 0$ implies $w^2 = u^2 + v + 2$. On the other hand, we have $u^2 + v^2 + w^2 = 1$, and hence we have $2w^2 = 1$, so that $w^2 = \frac{1}{2}$ and hence the positive number w must be $\frac{1}{2}\sqrt{2}$. Note that we must also have $u^2 + v^2 = \frac{1}{2}$.

Next, we note that the second coordinate is zero if and only if

$$u(w^2 - u^2) + uv^2 \quad = \quad 0$$

which means that either (a) we have $u = 0$, or else we have (b) $v^2 = u^2 - w^2$.

If (a) is true, then $v^2 = w^2 = \frac{1}{2}$, so that $v = \pm\frac{1}{2}\sqrt{2}$.

If (b) is true, then $u^2 + v^2 = w^2$ implies $v^2 = w^2 - u^2$, so that

$$u^2 - w^2 \quad = \quad v^2 \quad = \quad w^2 - u^2$$

and since the expressions on the left and right are negatives of each other it follows that $v^2 = -v^2$, so that $v = 0$. As in the preceding paragraph we then have $u^2 = w^2 = \frac{1}{2}$, so that $u = \pm\frac{1}{2}\sqrt{2}$.

Conversely, one can check directly that the cross product $\sigma_u \times \sigma_v$ vanishes at the four points $(u, v) = (0, \pm\frac{1}{2}\sqrt{2})$ and $(u, v) = (\pm\frac{1}{2}\sqrt{2}, 0)$, which correspond to the four points $p(0, \pm\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2})$ and $p(\pm\frac{1}{2}\sqrt{2}, 0, \frac{1}{2}\sqrt{2})$.

One can analyze the images of h_2 and h_3 similarly by simply permuting the coordinates in $\mathbf{R}^3$. In particular, the points in the image of h_2 for which g is not an immersion are given by $p(0, \frac{1}{2}\sqrt{2}, \pm\frac{1}{2}\sqrt{2})$ and $p(\pm\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}, 0)$, and particular, the points in the image of h_3 for which g is not an immersion are given by $p(\frac{1}{2}\sqrt{2}, \pm\frac{1}{2}\sqrt{2}, 0)$ and $p(\frac{1}{2}\sqrt{2}, , 0 \pm \frac{1}{2}\sqrt{2}, 0)$. Since $p(x, y, z) = p(-x, -y, -z)$, it follows that the map g is qn immersion except at the six distinct points listed below:

$$p(\tfrac{1}{2}\sqrt{2}, \pm\tfrac{1}{2}\sqrt{2}, 0)$$

$$p(\tfrac{1}{2}\sqrt{2}, 0, \pm\tfrac{1}{2}\sqrt{2})$$

$$p(0, \tfrac{1}{2}\sqrt{2}, \pm\tfrac{1}{2}\sqrt{2})$$

This completes the solution to the exercise.■