Preface

Formally, this course is a continuation of Mathematics 246A. However, it is not necessary to have taken the latter in order to start with this course for several reasons. First of all, we shall begin by taking a different approach to algebraic topology which does not immediately require knowledge of the constructions from 246A. There will be clear indications when we need input from that course, and references will be given. The somewhat different approach may shed some light on the abstract constructions that were necessary in 246A, and with hindsight the reasons for some of them may become more apparent.

We shall begin with a review *differential forms*, which are ordinarily covered in Mathematics 205C. One goal of the course is to give a topological explanation of the difference between closed and exact differential forms. Everything will start slowly, and at the beginning we shall consider the special case of interpreting the difference between the two types of differential forms in terms of the fundamental group. There is considerable overlap between these results and the Cauchy-Goursat Theorem in the theory of functions of a complex variable.

One reason for beginning slowly is to provide some opportunity for simultaneous review of some topics in 246A. More will be said about this in the main part of the course notes, but here is a start.

REVIEW SUGGESTION. This might be a good time to review the Preface from the 246A notes. The latter are available online at the following source:

`http://math.ucr.edu/∼res/math246A/algtopnotes.pdf`

This is probably also a good point to give the reference for the official course text.

> **A. Hatcher.** *Algebraic Topology* (Third Paperback Printing), *Cambridge University Press, New York NY*, 2002. ISBN: 0–521–79540–0.

This book can be legally downloaded from the Internet at no cost for personal use, and here is the link to the online version:

`www.math.cornell.edu/∼hatcher/AT/ATpage.html`

At some points we shall follow the text closely, but at others we shall go in different directions. The latter applies particularly to our discussion of algebraic topology and differential forms. One background reference for the latter is the following textbook for 205C:

> **L. Conlon.** Differentiable Manifolds. (Second Edition), *Birkhäuser-Boston, Boston MA*, 2001, ISBN 0–8176–4134–3.

Throughout the course we shall also use the following textbook for 205A and 205B as a reference for many topics and definitions:

J. R. Munkres. *Topology* (Second Edition), *Prentice-Hall, Saddle River NJ*, 2000. ISBN: 0–13–181629–2.

Content and objectives of this course

One of the central ideas in 246A was the definition of abelian groups called *homology groups* whose algebraic structure reflects many of the topological properties of a space. In particular, one can roughly think of the elements of homology groups as suitably defined equivalence classes of "nice" subspaces like closed curves or closed surfaces or, more generally, compact unbounded k-manifolds for suitable choices of k. An equally central idea in this course is the definition of **cohomology groups**, where as usual in mathematics the prefix **co-** indicates some sort of dual construction (say like the construction of a dual space associated to a vector space over a field). It is often convenient to view elements of cohomology groups as suitably defined equivalence classes of *measurement data* on the "nice" subspaces which represent elements of homology groups.

EXAMPLE. Suppose that $\omega = P\,dx + Q\,dy$ is a differential 1-form on an open connected subset U of $\mathbf{R}^2$ which is *closed* in the sense that its partial derivatives satisfy $P_y = Q_x$. Elements of the 1-dimensional homology of U are given by suitably defined equivalence classes of closed curves (specifically, one takes the unique maximal abelian quotient group of $\pi_1(U, u)$ for some arbitrary basepoint u). Given a smooth curve Γ we know how to define the line integral $\int_\Gamma \omega$, and it turns out that one can extend this definition to nonsmooth curves in a halfway reasonable manner. This line integral can then be viewed as defining measurement data for all closed curves, and it determines a 1-dimensional cohomology class of U. This class turns out to be zero if ω is an **exact** differential 1-form, but otherwise it might be nonzero.

In the Preface to the 246A notes we gave a few examples to indicate how the material from that course could shed some light on questions of independent interest. Here is a corresponding discussion for the present course:

1. One primary goal will be to give a unified approach to certain results in multivariable calculus involving the ∇ operator, Green's Theorem, Stokes' Theorem and the Divergence Theorem, and to formulate analogs for higher dimensions.

2. A related goal will be to provide a topological interpretation of certain consequences of the theorems mentioned in the preceding sentence. For example, if U is an open subset of $\mathbf{R}^2$ and ω is a closed 1-form on U as described above, then experience shows that there are often not too many possibilities for the value of the line integral $\int_\Gamma \omega$ where Γ ranges over all closed piecewise smooth curves in U. In particular, if U is the complement of a finite set of points, the results of this course will show that there are only countably many possible values for such line integrals.

3. Here is a question in a much different direction. On the torus $T^2 = S^1 \times S^1$ there are two standard closed curves given by the images of $\{1\} \times S^1$ and $S^1 \times \{1\}$. Of course, these curves meet in a single point, and one might ask if it is possible to find closed curves γ_1 and γ_2 which are (non-basepoint-preservingly) homotopic to these curves such that the images of γ_1 and γ_2 are disjoint. Experimentation suggests this is not possible, and the results obtained in this course will yield a mathematical proof that one can never find such curves.

4. In the preceding course we mentioned that a major goal of algebraic topology is to provide a setting for analyzing the set of homotopy classes of mappings $[X, Y]$ from one "nice space" X to a second "nice" space Y. The methods of Mathematics 205B and 246A yield a positive result in one simple but important case; specifically, if $Y = S^1$ and X is suitably restricted, then there is a canonical monomorphism from $[X, S^1]$ to the abelian group

$$\mathrm{Hom}\,(\,\pi_1(X, x_0),\ \mathbf{Z})$$

of algebraic homomorphisms from $\pi_1(X, x_0)$ to the infinite cyclic group $\mathbf{Z}$ (where x_0 is a suitably chosen basepoint), and it is given by taking a map $f : X \to S^1$ to the algebraic homomorphism from $\pi_1(X, x_0)$ to $\pi_1(S^1, 1) \cong \mathbf{Z}$ determined by the basepoint preserving map $f \cdot f(x_0)^{-1}$, where the raised dot indicates the usual complex multiplication on $S^1 \subset \mathbf{C}$. Furthermore, if X has a finite cell complex structure in the sense of 246A, then this monomorphism is an isomorphism (see the online document `bruschlinsky.pdf` for details and an alternate formulation in terms of results from 246A). One goal of the present course is to give a partial analog of these results if one replaces S^1 by S^n.

5. A more general issue involves the applicability of concepts from category theory. Algebraic topology is one branch of mathematics in which such ideas have been used successfully to solve questions of independent interest and to discover new and important phenomena. One objective of the course is to describe a few of these discoveries.

The keyed outline for the course (`math246Bkeyedoutline.pdf`) is very ambitious, and it is likely that not everything in the final units of the outline outline can be covered. However, priority will be given to two topics with ties to 246A; namely, the Alexander Duality Theorem — which places the classical Jordan-Brouwer separation theorems into a more general setting — and the Lefschetz Fixed Point Theorem — which may be viewed as a generalization of the Brouwer Fixed Point Theorem to spaces that are not contractible.

Background and review

This is a good time to read the section of the 246A notes called "Prerequisites." In this course we shall also use material from 205C when necessary, but most of the material will involve differential forms on open subsets of $\mathbf{R}^n$ for some n. A revised and slightly corrected version of a handout for this topic from Mathematics 205C

`extforms2007.ps`

is available in the course directory; the third section can be skipped on first reading because it only plays a limited role in the present course. Also, the following multivariable calculus textbook may be useful in connection with some examples we shall discuss:

> **J. E. Marsden and A. J. Tromba.** *Vector Calculus* (Fifth Edition), *W. H. Freeman & Co., New York NY*, 2003. ISBN: 0–7147–4992–0.

Portions of this course deal with the interaction between algebraic topology and other branches of mathematics, so it will be necessary to make some compromises in order to cover everything. Since the main emphasis of the course is on algebraic topology, we shall do this by varying the level of coverage; specifically, we shall try to make the topological content self-contained (at least when combined with the basic course references), but when we are considering interactions with other parts of mathematics, we shall sometimes assume whatever is needed from such areas.

I. Differential Forms and their Integrals

The purpose of this unit is to continue the discussion of differential forms from 205C in several directions. One objective is to prove a version of the result stating that the line integral of a closed 1-form over a closed curve only depends upon the homotopy class of the curve in the open set U on which the 1-form and curve are defined. A second objective is to relate the definition of integrals from 205C to the sorts of constructions one sees in multivariable calculus courses, and the third objective is to give a generalization of classical results in vector analysis (like Stokes' Theorem) to arbitrary dimensions.

I.1 : Differential 1–forms and the fundamental group

(Conlon, §§ 6.2 – 6.4)

In multivariable calculus one learns that certain line integrals in the plane of the form

$$\int_\Gamma P\,dx + Q\,dy$$

depend only on the endpoints of Γ. More precisely, if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

and P and Q have continuous partial derivatives on a **convex** open set, then the integral does not depend upon the path. In contrast, if we consider the line integral

$$\int_\Gamma \frac{x\,dy - y\,dx}{x^2+y^2}$$

over the counterclockwise unit circle $(\cos t, \sin t)$ for $0 \leq t \leq 2\pi$, then direct computation shows that the value obtained is 2π, but if we consider the corresponding line integral over the counterclockwise circle of radius $\frac{1}{3}$ centered at $(\frac{2}{3}, 0)$ with parametrization

$$x(t) = \tfrac{2}{3} + \tfrac{1}{3}\cos t, \qquad y(t) = \tfrac{1}{3}\sin t \qquad (0 \leq t \leq 2\pi)$$

then direct computation shows that the integral's value is zero. Since both of the curves we have described start and end at $(1, 0)$, obviously the line integral does depend upon the path in this case. It is natural to ask the extent to which the line integral does vary with the choice of path; the main result here states that the value depends only on the homotopy class of the path, where it is assumed that the homotopy keeps the endpoints fixed.

MAIN RESULT. *Let U be an open subset of the coordinate plane, let $P(x, y)$ and $Q(x, y)$ be two functions with continuous partials satisfying the previous condition on partial derivatives, and* *let Γ and Γ' be two piecewise smooth curves in U with the same endpoints such that Γ and Γ' are homotopic by an endpoint preserving homotopy. Then*

$$\int_\Gamma P\,dx + Q\,dy = \int_{\Gamma'} P\,dx + Q\,dy\ .$$

In particular, if Γ and Γ' are closed curves, then the line integrals agree if Γ and Γ' determine the same element of $\pi_1(U, \{\text{endpoint}\})$.

At the end of this section we shall explain how this result yields a complete description of all values that the line integral

$$\int_\Gamma P\,dx + Q\,dy = \int_{\Gamma'} P\,dx + Q\,dy$$

can take, where P and Q are the specific functions above and Γ is a closed piecewise smooth curve whose image lies in $\mathbf{R}^2 - \{(0,0)\}$.

Relation to the Cauchy-Goursat Theorem. If f is an analytic function of a complex variable on the open set $U \subset \mathbf{R}^2 = \mathbf{C}$ and we write $f = u + i\,v$ as usual, then for every piecewise smooth curve Γ we have

$$\int_\Gamma f(z)\,dz \quad = \quad \int_\Gamma u\,dx \ - \ v\,dy \ + \ i \cdot \int_\Gamma v\,dx \ + \ u\,dy$$

and by the Cauchy-Riemann Equations the integrands of the two summands satisfy the previously formulated condition $P_y = Q_x$, and hence if u and v are known to have continuous partial derivatives then the main result proves a reasonably good form of the Cauchy-Goursat Theorem. Since the usual definition of analytic function does not include the continuity assumption for the partial derivatives, the main result does not quite prove the entire Cauchy-Goursat Theorem, but it is possible to modify the argument slightly in order to obtain the general result in which one does not assume the partial derivatives are continuous (if one continues to develop the subject of complex variables, it turns out that u and v always have continuous partial derivatives, but this requires additional work).

The following alternate version of the Main Theorem is frequently found in books on multivariable calculus.

ALTERNATE STATEMENT OF MAIN THEOREM. *If P and Q are as above, U is a connected region, and Γ is a piecewise smooth closed curve that is homotopic to a constant in U, then*

$$\int_\Gamma P\,dx + Q\,dy \quad = \quad 0\ .$$

This follows immediately from the Main Result, observation (3) above, and the triviality of the fundamental group of U. Conversely, the Main Result follows from the Alternate Statement. To see this, in the setting of the Main Result the curve $\Gamma' + (-\Gamma)$ is a closed piecewise smooth curve that is homotopic to a constant (verify this!), so the Alternate Statement implies that the line integral over this curve is zero. On the other hand, this line integral is also the difference of the line integrals over Γ' and Γ. Combining these observations, we see that the line integrals over Γ' and Γ must be equal.

In fact, it will be more convenient for us to prove the Alternate Statement in the discussion below.

The next result is often also found in multivariable calculus texts.

COROLLARY. *If in the setting of the Main Result and its Alternate Statement we also know that the region U is simply connected then*

(*i*) *for every piecewise smooth closed curve Γ in U we have*

$$\int_\Gamma P\,dx \;+\; Q\,dy \quad = \quad 0$$

(*ii*) *for every pair of piecewise smooth curves Γ, Γ' with the same endpoints we have*

$$\int_\Gamma P\,dx \;+\; Q\,dy \quad = \quad \int_{\Gamma'} P\,dx \;+\; Q\,dy \; .$$

The first part of the corollary follows from the triviality of the fundamental group of U, the Alternate Statement of the Main Result, and the triviality of line integrals over constant curve. The second part follows formally from the first in the same way that the Main Result follows from its Alternate Statement.

Background from multivariable calculus

As noted above, the following result can be found in most multivariable calculus textbooks.

PATH INDEPENDENCE THEOREM. *Let U be a rectangular open subset of the coordinate plane of the form $(a_1, b_1) \times (a_2, b_2)$, let P and Q be functions with continuous partials on U such that*

$$\frac{\partial P}{\partial y} \quad = \quad \frac{\partial Q}{\partial x}$$

and let Γ and Γ' be two piecewise smooth curves in U with the same endpoints. Then

$$\int_\Gamma P\,dx + Q\,dy \quad = \quad \int_{\Gamma'} P\,dx + Q\,dy \; .$$

The underlying idea behind the proof is to construct a function f such that $\nabla f = (P, Q)$. Green's Theorem plays major role in showing that the partials of f have the desired values.

Notational and abuse of language conventions. Given two points $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$ in the coordinate plane, the closed straight line segment joining them is the curve $[\mathbf{p}, \mathbf{q}]$ with parametrization

$$x(t) \quad = \quad tp_1 + (1-t)p_2, \qquad y(t) \quad = \quad tq_1 + (1-t)q_2 \qquad (0 \leq t \leq 1) \; .$$

We would also like to discuss broken line curves, say joining $\mathbf{p}_0$ to $\mathbf{p}_1$ by a straight line segment, then joining $\mathbf{p}_1$ to $\mathbf{p}_2$ by a straight line segment, and so on. The points $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, etc. are called the *vertices* of the broken line curve. One technical problem with this involves the choices of linear parametrizations for the pieces. However, since line integrals for such curves do not depend upon such parametrizations and in fact we have

$$\int_C P\,dx + Q\,dy \quad = \quad \sum_i \int_{[\mathbf{p}_{i-1},\mathbf{p}_i]} P\,dx + Q\,dy$$

we shall not worry about the specific choice of parametrization. Filling in the details will be left as an exercise to a reader who is interested in doing so; this is basically elementary but tedious.

Integrals over broken line inscriptions

First some standard definitions. A *partition* of the interval $[a, b]$ is a sequence of points

$$\Delta : a \quad = \quad t_0 \quad < \quad t_1 \quad < \quad \cdots \quad < \quad t_m \quad = \quad b$$

and the *mesh* of Δ, written $|\Delta|$, is the maximum of the differences $t_i - t_{i-1}$ for $1 \leq i \leq m$. Given a piecewise smooth curve Γ defined on $[a, b]$, the *broken line inscription* $\mathbf{Lin}\,(\Gamma, \Delta)$ is the broken line curve with vertices

$$\Gamma(a) \quad = \quad \Gamma(t_0), \Gamma(t_1), \cdots \Gamma(t_m) \quad = \quad \Gamma(b)\ .$$

We are now ready to prove one of the key technical steps of the proof of the main result.

LEMMA. *Let U, be a connected open subset of $\mathbf{R}^2$, and let P, Q and Γ be as usual, where Γ is defined on $[a, b]$ and P and Q satisfy the condition*

$$\frac{\partial P}{\partial y} \quad = \quad \frac{\partial Q}{\partial x}\ .$$

Then there is a positive constant $\delta > 0$ such that for all partitions Δ of $[a, b]$ with $|\Delta| < \delta$ we have

$$\int_\Gamma P\,dx + Q\,dy \quad = \quad \int_{\mathbf{Lin}\,(\Gamma,\Delta)} P\,dx + Q\,dy\ .$$

Proof. If K is the image of Γ then K is a compact subset of the open set U, and therefore there is an $\varepsilon > 0$ so that if $\mathbf{x} \in \mathbf{R}^2$ satisfies $|\mathbf{x} - \mathbf{v}| < \varepsilon$ for some $\mathbf{v} \in K$ then $\mathbf{x} \in U$. It follows that if $\mathbf{v} \in K$ then the inner region for the square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon\sqrt{2}$ lies entirely in U.

By uniform continuity there is a $\delta > 0$ so that if $s, t \in [a, b]$ satisfy $|s - t| < \delta$ then

$$|\Gamma(s) - \Gamma(t)| \quad < \quad \frac{\varepsilon\sqrt{2}}{2}\ .$$

Let Δ be a partition of $[a, b]$ whose mesh is less than δ. Then for all i the restriction of Γ to $[t_{i-1}, t_i]$ lies in the open disk of radius $\frac{1}{2}\varepsilon\sqrt{2}$. It follows that both this restriction and the closed straight line segment joining $\Gamma(t_{i-1})$ to $\Gamma(t_i)$ lie in the open square region centered at $\Gamma(t_{i-1})$ with sides parallel to the coordinate axes of length of length $\varepsilon\sqrt{2}$; since the latter lies entirely in U. it follows that P and Q are defined on this square region. Therefore, by the previously quoted result from multivariable calculus we have

$$\int_{\Gamma|[t_{i-1},t_i]} P\,dx + Q\,dy \quad = \quad \int_{[\Gamma(t_{i-1}),\Gamma(t_i)]} P\,dx + Q\,dy$$

for each i. But the line integral over Γ is the sum of the line integrals over the curves $\Gamma|[t_{i-1}, t_i]$, and the line integral over the broken line inscription is the sum of the line integrals over the line segments $[\Gamma(t_{i-1}), \Gamma(t_i)]$, and therefore it follows that the line integral over Γ is equal to the line integral over the broken line inscription, as required.

Proof of the Alternate Statement of the Main Result

We may as well assume that Γ is defined on the unit interval $[0, 1]$ since we can always arrange this by a linear change of variables. Let $H : [0, 1] \times [0, 1] \to U$ be a continuous map such that $H(s, 0) = \Gamma(s)$ for all s and H is constant on both $[0, 1] \times \{1\}$ and $\{0, 1\} \times [0, 1]$.

If L is the image of H then L is a compact subset of the open set U, and as in the proof of the lemma there is an $\varepsilon' > 0$ so that if $\mathbf{x} \in \mathbf{R}^2$ satisfies $|\mathbf{x} - \mathbf{v}| < \varepsilon'$ for some $\mathbf{v} \in L$ then $\mathbf{x} \in U$. It follows that if $\mathbf{v} \in L$ then the inner region for the square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon'\sqrt{2}$ lies entirely in U.

By uniform continuity there is a $\delta' > 0$ so that if $\mathbf{s}, \mathbf{t} \in [0, 1] \times [0, 1]$ satisfy $|\mathbf{s} - \mathbf{t}| < \delta'$ then

$$|H(\mathbf{s}) - H(\mathbf{t})| \quad < \quad \frac{\varepsilon'\sqrt{2}}{2} \ .$$

Without loss of generality we may assume that δ' is no greater than the δ in the previous lemma. Let Δ be a partition of $[a, b]$ whose mesh is less than $\frac{1}{2}\delta'\sqrt{2}$, and choose a positive integer N such that

$$\frac{1}{N} \quad < \quad \frac{\delta'\sqrt{2}}{2} \ .$$

Then for all i such that $1 \le i \le m$ and all j such that $1 \le j \le N$ the restriction of H to $[t_{i-1}, t_i] \times \left[\frac{j-1}{N}, \frac{j}{N}\right]$ lies in an open disk of radius $\frac{1}{2}\varepsilon'\sqrt{2}$.

A special case. To motivate the remainder of the argument, we shall first specialize to the case where H extends to a map on an open set containing the square $[0, 1] \times [0, 1]$ and has continuous partials on this open set. For each i such that $0 \le i \le m$ and each j such that $1 \le j \le N$ let $A(i, j)$ be the broken line curve in the square with vertices

$$\left(0, \tfrac{j-1}{N}\right), \ \ldots \ \left(t_i, \tfrac{j-1}{N}\right), \left(t_i, \tfrac{j}{N}\right), \ \ldots \ \left(1, \tfrac{j}{N}\right) \ .$$

In other words, this curve is formed by starting with a horizontal line segment from $\left(0, \frac{j-1}{N}\right)$ to $\left(t_i, \frac{j-1}{N}\right)$, then concatenating with a vertical line segment from $\left(t_i, \frac{j-1}{N}\right)$ to $\left(t_i, \frac{j}{N}\right)$, and finally concatenating with a horizontal line segment from $\left(t_i, \frac{j}{N}\right)$ to $\left(1, \frac{j}{N}\right)$. If $W(i, j)$ denotes the composite $H \circ A(i, j)$, then it follows that $W(i, j)$ is a piecewise smooth closed curve in U. Furthermore, $W(m, 1)$ is just the concatenation of Γ with a constant curve and $W(0, N)$ is just a constant curve, so the proof of the main result reduces to showing that the line integrals of the expression

$P\,dx + Q\,dy$ over the curves $W(m,1)$ and $W(0,N)$ are equal. We claim this will be established if we can show the following hold for all i and j:

(1) The corresponding line integrals over the curves $W(0, j-1)$ and $W(m,j)$ are equal.

(2) The corresponding line integrals over the curves $W(i-1,j)$ and $W(i,j)$ are equal.

To prove the claim, first note that (2) implies that the value of the line integral over $W(i,j)$ is a constant z_j that depends only on j, and then note that (1) implies $z_{j-1} = z_j$ for all j. Thus the two assertions combine to show that the line integrals over all the curves $W(i,j)$ have the same value.

We begin by verifying (1). Since H is constant on $\{0,1\} \times [0,1]$, it follows that $W(m,j)$ is formed by concatenating $H|[0,1] \times \{\frac{j}{m}\}$ and a constant curve (in that order), while $W(0, j-1)$ is formed by concatenating a constant curve and $H|[0,1] \times \{\frac{j}{m}\}$ (again in the given order). Thus the line integrals over both $W(0,j-1)$ and $W(m,j)$ are equal to the line integral over $H|[0,1] \times \{\frac{j}{m}\}$, proving (1).

Turning to (2), since the broken line curves $A(i,j)$ and $A(i-1,j)$ differ only by one vertex, it follows that the difference

$$\int_{W(i,j)} P\,dx \;+\; Q\,dy \;-\; \int_{W(i-1,j)} P\,dx \;+\; Q\,dy$$

is equal to

$$\int_{V(i,j)} P\,dx \;+\; Q\,dy \;-\; \int_{V'(i,j)} P\,dx \;+\; Q\,dy$$

where $V(i,j)$ is the composite of H with the broken line curve with vertices

$$\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad \left(t_i, \tfrac{j-1}{N}\right), \quad \left(t_i, \tfrac{j}{N}\right)$$

and $V'(i,j)$ is the composite of H with the broken line curve with vertices

$$\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad \left(t_{i-1}, \tfrac{j}{N}\right), \quad \left(t_i, \tfrac{j}{N}\right) \ .$$

Our hypotheses imply that both of these curves lie in an open disk of radius $\frac{1}{2}\varepsilon'\sqrt{2}$ and thus also in the open square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon'\sqrt{2}$; by construction the latter lies entirely in U. Therefore by the previously quoted result from multivariable calculus we have

$$\int_{V(i,j)} P\,dx + Q\,dy \quad = \quad \int_{V'(i,j)} P\,dx + Q\,dy$$

for each i and j, so that the difference of the line integrals vanishes. Since this difference is also the difference between the line integrals over $W(i,j)$ and $W(i-1,j)$, it follows that the line integrals over the latter two curves must be equal.

The general case. If H is an arbitrary continuous function the preceding proof breaks down because we do not know if the continuous curves $W(i,j)$ are well enough behaved to define line integrals. We shall circumvent this by using broken line approximations to these curves and appealing to the previous lemma to relate the value of the line integrals over these approximations to the value on the original curve. Since the proof is formally analogous to that for the special case we shall concentrate on the changes that are required.

Let $X(i,j)$ denote the broken line curve with vertices

$$H\left(0,\tfrac{j-1}{N}\right),\ldots H\left(t_i,\tfrac{j-1}{N}\right),H\left(t_i,\tfrac{j}{N}\right),\ldots H\left(1,\tfrac{j}{N}\right)\ .$$

By our choice of Δ these broken lines all lie in U, and the constituent segments all lie in suitably small open disks inside U.

We claim that it will suffice to prove that the line integrals over the curves $X(0,j-1)$ and $X(m,j)$ are equal for all j and for each j the corresponding line integrals over the curves $X(i-1,j)$ and $X(i,j)$ are equal. As before it will follow that the line integrals over all the broken line curves $X(i,j)$ have the same value. But $X(m,N)$ is a constant curve, so this value is zero. On the other hand, by construction the curve $X(m,1)$ is formed by concatenating $\mathbf{Lin}\,(\Gamma,\Delta)$ and a constant curve, so this value is also the value of the line integral over $\mathbf{Lin}\,(\Gamma,\Delta)$. But now the Lemma implies that the values of the corresponding line integrals over Γ and $\mathbf{Lin}\,(\Gamma,\Delta)$ are equal, and therefore the value of the line integral over the original curve Γ must also be equal to zero.

The first set of equalities follow from the same sort argument used previously for $W(0,j-1)$ and $W(m,j)$ with the restriction of Γ replaced by the broken line curve with vertices

$$H\left(0,\tfrac{j}{N}\right),\ldots H\left(1,\tfrac{j}{N}\right)\ .$$

To verify the second set of equalities, note that the difference between the values of the line integrals over $X(i,j)$ and $X(i-1,j)$ is given by

$$\int_{C(i,j)} P\,dx\ +\ Q\,dy\ -\ \int_{C'(i,j)} P\,dx\ +\ Q\,dy$$

where $C(i,j)$ is the broken line curve with vertices

$$H\left(t_{i-1},\tfrac{j-1}{N}\right),\quad H\left(t_i,\tfrac{j-1}{N}\right),\quad H\left(t_i,\tfrac{j}{N}\right)$$

and $C'(i,j)$ is the broken line curve with vertices

$$H\left(t_{i-1},\tfrac{j-1}{N}\right),\quad H\left(t_{i-1},\tfrac{j}{N}\right),\quad H\left(t_i,\tfrac{j}{N}\right).$$

By the previously quoted result from multivariable calculus we have

$$\int_{C(i,j)} P\,dx + Q\,dy \quad = \quad \int_{C'(i,j)} P\,dx + Q\,dy$$

for each i and j, and therefore the difference between the values of the line integrals must be zero. Therefore the difference between the values of the line integrals over $X(i,j)$ and $X(i-1,j)$ must also be zero, as required. This completes the proof.■

An example

Suppose now that $U = \mathbf{R}^2 - \{\mathbf{0}\}$ and ω is the closed 1-form

$$\frac{x\,dy \;-\; y\,dx}{x^2 \;+ y^2}\;.$$

We would like to describe the possible values for the line integral

$$\int_\Gamma \omega$$

as Γ ranges over all closed piecewise smooth curves in U.

We shall first consider curves of this type whose initial and final value is the unit vector $\mathbf{e}_1 = (1,0)$, Since we have

$$\pi_1(S^1, \mathbf{e}_1) \;\cong\; \pi_1(U, \mathbf{e}_1) \;\cong\; \mathbf{Z}$$

and the results of this section show that the value of the line integral only depends upon the class of Γ in the fundamental group, it follows that there are only countably many possible values for the line integral. Furthermore, by the definition of concatenation for curves we have

$$\int_{\Gamma+\Phi} \omega \;\;=\;\; \int_\Gamma \omega \;+\; \int_\Phi \omega$$

it follows that in fact the line integral construction yields a homomorphism from $\mathbf{Z}$ to $\mathbf{R}$. Therefore it is enough to evaluate the line integral on a curve which generates the fundamental group. Of course, the standard generator is the counterclockwise circle

$$\Psi(t) \;\;=\;\; (\cos 2\pi\, t,\; \sin 2\pi\, t)$$

and by a standard exercise in multivariable calculus the value of $\int_\Psi \omega$ in this case is 2π. Therefore we have the following:

The set of all possible values for the line integral $\int_\Gamma \omega$ must be the set of all integral multiples of 2π.

If the initial and final point of a curve is some point $\mathbf{p}$ which is not necessarily $\mathbf{e}_1$, we may retrieve the same conclusion as follows: Let α be a piecewise smooth curve joining $\mathbf{e}_1$ to $\mathbf{p}$. Then we know that the construction sending a closed curve Γ based at $\mathbf{e}_1$ to the closed curve

$$\Gamma^* \;\;=\;\; -\alpha \;+\; \Gamma \;+\; \alpha$$

passes to an isomorphism of groups from $\pi_1(U, \mathbf{e}_1)$ to $\pi_1(U, \mathbf{p})$. Since the line integrals of Γ^* and Γ are equal (why?), it follows that the images of the associated homomorphisms from $\pi_1(U, \mathbf{e}_1)$ and $\pi_1(U, \mathbf{p})$ are also equal, so that the latter also consists of all integral multiples of 2π.$\blacksquare$

I.2 : Extending Green's and Stokes' Theorems

(Conlon, § 8.1)

In this section we shall use the contents of `extforms2007.pdf` as needed (with the exception of Section 3 in the latter). This is probably also a good time to look back at Sections I.2 from the 246A and also Section III.2 up to Lemma 1 of the latter. Illustrations for this section appear in the following separate file:

`http://math.ucr.edu/~res/figures0102.pdf`

Objectives

In advanced calculus textbooks, it is easy to find proofs of basic results in vector analysis like *Green's Theorem*, *Stokes' Theorem*, and the *Divergence Theorem* in special cases. For example, it is very easy to derive Green's Theorem in the case of regions defined by standard systems of inequalities

$$a \;\leq\; x \;\leq b, \qquad\qquad g(x) \;\leq\; y \;\leq\; f(x)$$

where g and f are continuous functions such that $g(x) \;<\; f(x)$, at least if $x \neq a,\; b$ (see the first illustration in `figures0102.pdf`). However, as noted in many (most?) advanced calculus texts, the result is true in far more general cases, including regions whose boundaries are given by several closed curves (the second illustration in `figures0102.pdf`). The goal of this section is to discuss some of the tools needed in order to extend the previously mentioned results in vector analysis from simple cases to more general ones.

Change of variables formulas

Most advanced calculus texts do not discuss the role of change of variables formulas in connection with the main theorems of vector analysis. Conceptually, the idea is clear. Suppose that we are given a closed region Ω in the plane whose boundary is given by several curves Γ_i with suitable senses of directions (there is an outermost curve which has a counterclockwise sense, and possibly inner curves which will each have a clockwise sense). Let T be a homeomorphism which is defined on an open set containing Ω such that the coordinate functions of T have continuous partial derivatives of all orders and the Jacobian is always positive. Then the image $T[\Omega]$ will be another region in the plane whose boundary consists of similar curves, each having the same sense as its inverse image (the positivity of the Jacobian is needed to ensure this condition).[1] As before, there are illustrations in `figures0102.pdf`).

[1] The assertion in parentheses depends upon knowing that the sense of a closed curve does not change if the Jacobian is positive and the sense is reversed if the Jacobian is negative. This can be seen fairly directly if T is an invertible linear transformation and Γ is the counterclockwise unit circle. Good examples to consider are the linear transformations $T(x, y) = (x + y, y)$ and $T(x, y) = (x, -y)$.

It is natural to ask whether one can prove directly that Green's Theorem holds for the transformed region and boundary curves if it is known to hold for the original region and boundary curves. In fact, this can be shown using the standard change of variables formula for double integrals and similar results for line integrals, but this is usually not done in advanced calculus texts, mainly because the computations needed to verify such a formula quickly become very messy. However, if one uses differential forms, one can do everything fairly easily as indicated below. To simplify the discussion we shall assume that the boundary of Ω consists of a single curve Γ.

The line integrals over the paths Γ and $T \circ \Gamma$ are related by a simple change of variables argument. In the language of `extforms2007.pdf`, the formula is

$$\int_{\Gamma} T^*\theta \quad = \quad \int_{T\circ\Gamma} \theta$$

where $\theta \;=\; P\,dx + Q\,dy$ and $T(u,v) \;=\; (x,y)$ (see the bottom of page 5 in the cited document). By the results of Section 4 in `extforms2007.pdf` translating Green's Theorem into a statement about differential forms, we can rewrite the left hand side as

$$\int_{\Omega} d\,(T^*\theta) \quad = \quad \int_{\Omega} T^*(d\,\theta)$$

where the equality of the terms follows from the Theorem near the bottom of page 5 in the cited reference. Yet another application of the Change of Variables formula below the statement of that theorem shows that the right hand side of the preceding result is equal to

$$\int_{T[\Omega]} d\,\theta$$

and if we combine all these equations, we see that Green's Theorem holds for the transformed curve and the transformed region.

An example

At this point we shall start using material on simplicial decompositions and simplicial complexes from Section I.2 of the 246A notes.

A standard approach to proving more general versions of Green's Theorem is to combine the change of variables principle with

(i) a nice decomposition of Ω into regions that can be analyzed using simple cases of Green's Theorem.

(ii) the change of variables principle described above.

For example, suppose that we are given the closed region bounded by a Star of David curve. It follows immediately that this closed region has a *simplicial decomposition* in the sense of Section I.2 in the 246A notes (see `figures0102.pdf`); specifically, we can cut the region up into solid triangular regions as illustrated in the picture such that the union of these solid regions is the original subset and the intersection of two solid triangular regions in the collection is either a common edge or a common vertex of the boundary triangles.

We can then apply Green's Theorem to each of the solid triangular regions, and thus the integral over the whole region is equal to the line integrals over the boundary curves of the solid triangular regions. However, one quickly sees that the line integrals over the "new" pieces of boundary curves — the pieces that were introduced when one cut up the original region — will cancel each other in pairs, so the sum of the line integrals over the boundary triangles will reduce to the line integral over the original Star of David curve.

Reformulation in terms of simplicial chains

Predictably, we shall be using material from the 246A notes on simplicial chains (Section III.3 of the 246A notes) in the discussion below. In addition, we shall need the following elaboration of the discussion in the first paragraph of Section IV.1 from the same notes:

LEMMA 0. *Let Λ_n be the n-simplex in $\mathbf{R}^n$ whose vertices are $\mathbf{0}, \mathbf{e}_1. \cdots, \mathbf{e}_n$ so that Λ is the set of all $(x_1, \cdots, x_n)$ satisfying $x_j \geq 0$ for all j and $\sum_j x_j \leq 1$. Let $\mathbf{v}_0, \cdots, \mathbf{v}_n \in \mathbf{R}^n$ be points that also are the vertices of an n-simplex $\mathbf{S}$. Then there is an affine map $T : \Lambda_n \to S$ which is $1-1$ onto, and for which the Jacobian of T at each point is positive.*

Proof. Let T_0 be the unique affine homeomorphism which sends $\mathbf{0}$ to $\mathbf{v}_0$ and $\mathbf{e}_i$ to $\mathbf{v}_i$ for all $i \geq 0$. Then T_0 has all the desired properties except perhaps the positivity of the Jacobian. In any case we know that the Jacobian is everywhere positive or everywhere negative. If the Jacobian is everywhere positive then we can simply take $T = T_0$. If not, let σ be the map from Λ_n to itself which switches coordinates, and take $T = T_0 \circ \sigma$; it will follow that T has all the required properties.■

In the spirit of Section IV.1 from the 246A notes we should also observe that there is a standard affine homomorphism between Λ_n and the standard simplex $\Delta_n \subset \mathbf{R}^{n+1}$ with vertices $\mathbf{v}_i = \mathbf{e}_{i+1}$. Specifically, take the affine map which sends $\mathbf{0}$ to $\mathbf{v}_0$ and $\mathbf{e}_i$ to $\mathbf{v}_i$ for $i > 0$.

We can relate the preceding constructions on the Star of David set to the constructions of the 246A notes as follows. As before, let $\Lambda_2 \subset \mathbf{R}^2$ be the solid triangular region with vertices $\mathbf{0} = (0,0)$, $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$, so that Λ_2 consists of all (x,y) such that $x, y \geq 0$ and $x + y \leq 1$. For each solid triangular region, or 2-*simplex*, α in the Star of David set, by Lemma 0 there is a 1–1 onto affine map T_α from Λ_2 onto α such that the Jacobian of T_α is always positive. As in Section IV.1 of the 246A notes, each T_α is naturally associated to a free generator of the ordered simplicial chain group $C_2(P)$, where P is the closed Star of David region. The double integral over P in Green's Theorem is a sum of the double integrals over the images of the mappings T_α. By the Change of Variables formulas, this may be viewed as a sum of double integrals over the standard 2-simplex Λ_2 corresponding to the changes of variables given by the maps T_α. Stretching the language still further, we can think of the original double integral as a sum of double integrals determined by the ordered simplicial chain $\sum_\alpha T_\alpha \in C_2(P)$.

Suppose we now apply Green's Theorem to each of the summands. For each summand the associated double integral is equal to a line integral over a path which traces the boundary. Now the definition of simplicial chains and boundaries associates to each 2-simplex a 1-chain called the boundary; specifically, if we let ∂_0, ∂_1 and $\partial 2$ be the straight line segment curves which go from $\mathbf{e}_1$

to $\mathbf{e}_2$, from $\mathbf{0}$ to $\mathbf{e}_2$ and $\mathbf{0}$ to $\mathbf{e}_1$ respectively (so that ∂_i gives the side opposite the i^{th} vertex), then the boundary 1-chain

$$d(\Lambda_2) \quad = \quad \partial_0 \;+\; (-\partial_1) \;+\; \partial_2$$

describes a parametrization for the boundary curve of Λ_2 in the counterclockwise sense, and hence we may view each double integral over Λ_2 as an appropriate line integral over the curve described by $d(\Lambda_2)$.

If we apply this to the chain $\sum_\alpha T_\alpha$, we see that the original double integral over the Star of David region is equal to the following sum of line integrals:

$$\sum_\alpha \int_{\partial_0 T_\alpha} \omega \;-\; \int_{\partial_1 T_\alpha} \omega \;+\; \int_{\partial_2 T_\alpha} \omega$$

Symbolically, we may view this as a line integral over the simplicial chain

$$d\left(\sum_\alpha T_\alpha\right) \quad \in \quad C_1(P) \; .$$

An inspection of the picture suggests that this chain can be simplified dramatically. Namely, the algebraic boundary consists only of a collection of straight line segments which define the topological boundary curve of the original region in the counterclockwise sense (the latter relies on our assumptions about positive Jacobians). All the extra straight line curves from the terms dT_α which do not lie entirely in the boundary turn out to cancel each other in pairs. This means that the line integral over the boundary chain described above must be equal to the line integral of the boundary curve for the original region, where as before the sense of the boundary is counterclockwise. This completes the derivation of Green's Theorem from the special case for triangular regions and the Change of Variables principle, at least for our example of the Star of David region. The same considerations work for an arbitrary region which has a simplicial decomposition and a boundary which is a simple closed curve as above.

In fact, similar considerations work for regions which have simplicial decompositions, but whose boundaries are unions of simple closed curves. For example, consider the example of a solid square with a square hole in the middle (see `figures0102.pdf` once more). Note that the boundary chain in the figure is a sum of two pieces, one of which corresponds to the outer boundary in the counterclockwise sense and the other of which corresponds to the inner boundary in the clockwise sense. As indicated by one of the drawings, if there are two inner curves, then the boundary splits into a sum of three pieces corresponding to the boundary curves; as before, the outer boundary has a counterclockwise sense and the inner boundary curves both have clockwise senses.

Green's Theorem in more general situations

A general region may have curves composed of pieces that are not straight line curves, and it will be necessary to replace the affine mappings and simplicial chains by more complicated objects. Specifically, we shall want maps T_α which are defined on open sets containing the 2-simplex Λ_2, with continuous partial derivatives and positive Jacobians at every point, and such that the restriction to Λ_2 will be 1–1. Furthermore, we want the images of these mappings to be nonoverlapping in the sense that the image of T_α meets the image of T_β in either (i) the image of a common boundary edge-curve, (ii) the image of a common vertex, and when the intersections are a common edge we want the common boundary curves to have compatible parametrizations. It may not be obvious that all this can be achieved, but in fact it is always possible to do so (however, the proof is definitely nontrivial), and illustrations appear in the file `figures0102.pdf`.

The resulting mappings T_α then have natural interpretations as generators of the group of **singular** 2-chains $\mathbf{S}_2(X)$ defined in Section IV.1 of the 246A notes; here X is the closed region that we are cutting into pieces). Recall that if n is a nonnegative integer, then the group of singular n-chains is a free abelian group with generating set given by continuous mappings T from the standard n-simplex Δ_n in $\mathbf{R}^{n+1}$ (its vertices are the standard unit vectors) to X; we may use the canonical affine homeomorphism from

$$\Lambda_n \quad = \quad \left\{ (x_1, \ \cdots \ , x_n) \in \mathbf{R}^n \mid x_i \ \geq \ 0 \ , \quad \sum_i \ x_i \ \ \leq \ \ 1 \right\}$$

to Δ_n sending $(x_1, \ \cdots \ , x_n)$ to

$$\left(1 \ - \ \sum_i \ x_i \right) \mathbf{e}_1 \ + \ \sum_j \ x_j \mathbf{e}_{j+1}$$

to identify singular n-simplices with continuous mappings defined on Λ_n. Thus the sum of the corresponding singular 2-simplices $\sum_\alpha \ T_\alpha$ may then be viewed as a singular 2-chain, and we can take its boundary using the same formula as in the simplicial case, obtaining an element in $\mathbf{S}_1(X)$. As in the preceding cases, it turns out that the boundary chain simplifies a sum of pieces $\sum_j \ \gamma_j$, where the γ_j's are a nonoverlapping collection of parametrizations for the boundary of X in the counterclockwise sense (and they give the entire boundary curve).

Stokes' Theorem and the Divergence Theorem

Similar considerations work in both cases. For Stokes' Theorem, one must take the smooth mappings T_α such that the normal directions given by

$$\frac{\partial T_\alpha}{\partial x} \times \frac{\partial T_\alpha}{\partial y}$$

are compatible in an appropriate sense. For the Divergence Theorem, there is an added complication of understanding what sorts of 3-dimensional building blocks should be used. It turns out that the right sorts of objects are given by images of the 3-simplex $\Lambda_3 \subset \mathbf{R}^3$ whose vertices are the zero vector and the three standard unit vectors; equivalently, Λ_3 is the set of all $(x, y, z) \in \mathbf{R}^3$ such that $x, y, z \geq 0$ and $x + y + z \leq 1$. Once again, the boundary formula from Section III.2 of the 246A notes is the right one for working with surface integrals over the boundaries.

Final remarks

Clearly the preceding discussion has not given many proofs. In fact, one needs methods from algebraic topology and further input from geometrical topology in order to give rigorous justifications of everything that is said here. Our purpose right now is mainly to provide motivation, giving important links between the algebraic theory of simplicial and singular chains in 246A and familiar issues related to evaluating line, surface and volume integrals by taking complicated sets and cutting them up into relatively simple pieces. Later in these notes we may say more about the machinery needed to give a rigorous justification of the general forms of Green's Theorem, Stokes' Theorem and the Divergence Theorem.

I.3 : Generalized Stokes' Formula

(Conlon, § 8.2)

The main purpose of this section is to strengthen the ties between differential forms from 205C and homological chains from 246A that were discussed in the previous section.

The combinatorial form of Stokes' Theorem (Theorem 8.2.9 on pages 251 – 255 of Conlon) is the fundamental link between the calculus of exterior differential forms and the singular homology theory that was defined and studied in 246A. In the preceding section we examined the 2-dimensional case in considerable detail. This section will concentrate on the most basic aspect of Stokes' Theorem in higher dimensions; namely, its validity in a fundamental special case. We shall be using the basic definitions for chains, differentials and homology in Units III and IV of the 246A notes extensively in this section.

Let q be a nonnegative integer. In 246A we defined a *singular q-simplex* in a topological space X to be a continuous mapping $T : \Delta_q \to X$, where Δ_q is the simplex in $\mathbf{R}^{q+1}$ whose vertices are the standard unit vectors; the group of singular q-chains $\mathbf{S}_q(X)$ was then defined to be the free abelian group on the set of singular q-simplices. The first step in this section is to is to define an analog of these groups involving *smooth* mappings if X is an open subset of $\mathbf{R}^n$ for some n.

Definition. Let q be a nonnegative integer, and as in the preceding section let $\Lambda_q \subset \mathbf{R}^q$ be the q-simplex whose vertices are $\mathbf{0}$ and the standard unit vectors. Also, let U be an open subset of $\mathbf{R}^n$ for some $n \geq 0$. A *smooth singular q-simplex* in U is a continuous map $T : \Lambda_q \to U$ which is *smooth* — in other words, there is some open neighborhood W_T of Λ_q in $\mathbf{R}^q$ such that T extends to a map $W_T \to U$ which is smooth in the usual sense (the coordinate functions have continuous partial derivatives of all orders). The group of *smooth singular q-chains* $\mathbf{S}_q^{\mathrm{smooth}}(U)$ is the free abelian group on all smooth singular q-simplices in U.

There is an obvious natural relationship between the smooth and ordinary singular chain groups which is given by the standard affine isomorphism φ from Δ_q to Λ_q defined on vertices by $\varphi(\mathbf{e}_1) = \mathbf{0}$ and $\varphi(\mathbf{e}_i) = \mathbf{e}_{i-1}$ for all $i > 1$. Specifically, each smooth singular q-simplex $T : \Lambda_q \to U$ determines the continuous singular q-simplex $T \circ \varphi : \Delta_q \to U$. The resulting map of singular chain groups will be denoted by

$$\varphi^{\#} : \mathbf{S}_q^{\mathrm{smooth}}(U) \longrightarrow \mathbf{S}_q(U)$$

with subscripts or superscripts added if it is necessary to keep track of q or U.

One important feature of the ordinary singular chain groups is that they can be made into a chain complex, and it should not be surprising to learn that there is a compatible chain complex structure on the groups of smooth singular chains. We recall the definition of the chain complex structure on $\mathbf{S}_*(X)$ for a topological space X, starting with the preliminary constructions. If Δ_q is the standard q-simplex, then for each i such that $0 \leq i \leq q$ there is an i^{th} face map $\partial_i : \Delta_{q-1} \to \Delta_q$ sending the domain to the face of Δ_q opposite the vertex $\mathbf{e}_{i+1}$ with $\partial_i(\mathbf{e}_j) = \mathbf{e}_j$ if $j \leq i$ and

$\partial_i(\mathbf{e}_j) \;=\; \mathbf{e}_{j+1}$ if $j \geq i+1$. Then each face map ∂_i defines function from singular q-simplices to singular $(q-1)$-simplices by the formula $\partial_i(T) \;=\; T \circ \partial_i$, and the formula

$$d_q \quad = \quad \sum_{i=0}^{q} (-1)^i \, \partial_i$$

defines a homomorphism from $\mathbf{S}_q(X)$ to $\mathbf{S}_{q-1}(X)$ with some important formal properties given by Theorem III.3.2 and the first two results in Section IV.1 of the 246A notes.

For the analogous constructions on smooth singular chain groups, we first need compatible face maps on Λ_q. Te simplest way to do this is to relabel the vertices of the latter as $\mathbf{0} = \mathbf{v}_0$ and $\mathbf{e}_i = \mathbf{v}_{i+1}$ for all i; then we may define ∂_i^{Λ} in the same way as ∂_i, the only difference being that we replace the vertices $\mathbf{e}_j$ for Δ_q by the vertices $\mathbf{v}_j$ for Λ_q.

We claim that if $T : \Lambda_q \to U$ is a smooth singular simplex then are all of the faces given by the composites $T \circ \partial_i^{\Lambda}$; this follows because each of maps ∂_i^{Λ} is an affine mapping and hence is smooth.

It follows immediately that the preceding constructions are compatible with the simplex isomorphisms φ constructed above, so that $\varphi^{\#} \circ \partial_i = \partial_i^{\Lambda} \circ \varphi^{\#}$, and if we define

$$d_q^{\mathrm{smooth}} : \mathbf{S}_q^{\mathrm{smooth}}(U) \;\longrightarrow\; \mathbf{S}_{q-1}^{\mathrm{smooth}}(U)$$

to be the sum of the terms $(-1)^i \partial_i^{\Lambda}$, then one has the following compatibility between smooth and singular chains.

PROPOSITION 1. *Let U be an open subset of $\mathbf{R}^n$ for some n, and let $\varphi^{\#} : \mathbf{S}_q^{\mathrm{smooth}}(U) \to \mathbf{S}_q(U)$ and d_*^{smooth} be the map given by the preceding constructions. Then the latter map makes $\mathbf{S}_*^{\mathrm{smooth}}(U)$ into a chain complex such that $\varphi^{\#}$ is a morphism of chain complexes.*

The assertion in the first sentence can be verified directly from the definitions, and the first assertion in the second sentence follows from the same sort of argument employed to prove Theorem III.3.2 in the 246A notes. Finally, the fact that $\varphi^{\#}$ is a chain complex morphism is an immediate consequence of the assertion in the first sentence and the definitions of the differentials in the two chain complexes in terms of the maps ∂_i and ∂_i^{Λ}.■

We shall denote the homology of the complex of smooth singular chains by $H_*^{\mathrm{smooth}}(U)$ and call the associated groups the *smooth singular homology groups* of the open set $U \subset \mathbf{R}^n$. In the next unit we shall prove the following fundamentally important result.

ISOMORPHISM THEOREM. *For all open subsets $U \subset \mathbf{R}^n$, the associated homology morphism $\varphi_*^{\#}$ from the smooth singular homology groups $H_*^{\mathrm{smooth}}(U)$ to the ordinary singular homology groups $H_*(U)$.*

Functoriality properties

In order to prove the Isomorphism Theorem, we need to establish additional properties of smooth singular chain and homology groups that are similar to basic properties of ordinary singular chain and homology groups. The first of these is a basic naturality property:

PROPOSITION 2. *Let $U \subset \mathbf{R}^n$, (etc.) be as above, let $V \subset \mathbf{R}^m$ be open, and let $f : U \to V$ be a smooth mapping from U to V (the coordinates have continuous partial derivatives of all orders). Then there is a functorial chain map $f_\#^{\mathrm{smooth}} : \mathbf{S}_*^{\mathrm{smooth}}(U) \to \mathbf{S}_*^{\mathrm{smooth}}(V)$ such that $f_\#^{\mathrm{smooth}}$ maps a smooth singular q-simplex T to $f \circ T$ and we have the naturality property*

$$f_\# \circ \varphi^\# \quad = \quad \varphi^\# \circ f_\#^{\mathrm{smooth}}$$

where $f_\#$ is the corresponding map of smooth singular chains from $\mathbf{S}_*(U)$ to $\mathbf{S}_*(V)$.■

COROLLARY 3. *In the setting of the preceding result, one has functorial homology homomorphisms on smooth singular homology, and the maps $\varphi_*^\#$ define natural transformations from smooth singular homology to ordinary singular homology.*■

Combining this with the Isomorphism Theorem mentioned earlier, we see that the construction $\varphi_*^\#$ determines a natural isomorphism from smooth singular homology to ordinary singular homology for open subsets of Euclidean spaces.

Since we are already discussing functoriality, this is a good point to mention some properties of this sort which hold for differential forms but were not formulated in `extforms2007.pdf`:

THEOREM 4. *Let $f : U \to V$ and $g : V \to W$ be smooth mappings of open subsets in Cartesian (Euclidean) spaces $\mathbf{R}^n$ where n need not be the same for any of U, V, W. Then the pullback construction on differential forms satisfies the identity $(g \circ f)^\# = f^\# \circ g^\#$. Furthermore, if f is the identity on U then $f^\#$ is the identity on $\wedge^*(U)$.*

The second of these is trivial, and the first is a direct consequence of the definitions and the Chain Rule for derivatives of composite maps.■

Integration over smooth singular chains

If U is an open subset of $\mathbf{R}^n$ and $T\Lambda_q \to U$ is a smooth singular q-simplex, then the basic integration formula in `extforms2007.pdf` provides a way of defining an integral $\int_T \omega$ if $\omega \in \wedge^q(U)$. There is a natural extension of this to singular chains; if $\mathbf{c}$ is the smooth singular chain $\sum_i n_i T_i$ where the n_i are integers, then since the group of smooth singular q-chains is free abelian on the smooth singular q-simplices the following is well defined:

$$\int_{\mathbf{c}} \omega \quad = \quad \sum_i n_i \int_{T_i} \omega$$

This definition has the following invariance property with respect to smooth mappings $f : U \to V$.

PROPOSITION 5. *Let $\mathbf{c} \in \mathbf{S}_q(U)$, where U is above, let $f : U \to V$ be smooth and let $\omega \in \wedge^q(V)$. Then we have*

$$\int_{f_\#^{\mathrm{smooth}}(\mathbf{c})} \omega \quad = \quad \int_{\mathbf{c}} f^\# \omega \ .$$

This follows immediately from the definition of integrals and the Chain Rule.■

The combinatorial form of the **Generalized Stokes' Formula** is a statement about integration of forms over smooth singular chains.

THEOREM 6. (Stokes' Formula, combinatorial version) *Let* $\mathbf{c}$, U, ω...(*etc.*) *be as above. Then we have*

$$\int_{d\mathbf{c}} \omega \quad = \quad \int_{\mathbf{c}} d\,\omega \ .$$

Full proofs of this result appear on pages 251–253 of Conlon and also on pages 272–275 of Rudin, *Principles of Mathematical Analysis* (3^{rd} Ed.). Here is an outline of the basic steps: First of all, by additivity it is enough to prove the result when $\mathbf{c}$ is given by a smooth singular simplex T. Next, by Proposition 5 and the identity $f^{\#} \circ d \;=\; d \circ f^{\#}$ (see `extforms2007.pdf` for this), we know that it suffices to prove the result when T is the universal singular simples $\mathbf{1}_q$ defined by the inclusion of Λ_q into some small open neighborhood W_0 of Λ_q. In this case the integrals reduce to ordinary integrals in $\mathbf{R}^q$. We can reduce the proof even further as follows: Let $\theta_i \in \wedge^{q-1}(W_0)$ be the basic $(q-1)$-form $dx^{i_1} \wedge \ \cdots \ dx^{i_{q-1}}$, where $i_1 < \ \cdots \ < i_{q-1}$ runs over all elements of $\{1, \ \cdots \ , q\}$ **except** i. By additivity it will suffice to prove Theorem 6 for $(q-1)$-forms expressible as $g\,\theta_i$, where g is a smooth function on W_0. Yet another change of variables argument shows that it suffices to prove the result for $(q-1)$-forms expressible as $g\,dx^2 \wedge \ \cdots \ \wedge dx^q$. Now the exterior derivative of the latter form is equal to

$$\frac{\partial g}{\partial x^1} \cdot dx^1 \wedge \ \cdots \ dx^q$$

so the proof reduces to evaluating the integral of the left hand factor in this expression over Λ_q, and this is done by viewing this multiple integral as an interated integral and applying the Fundamental Theorem of Calculus.$\blacksquare$

A "global" version of Stokes' Formula for arbitrary dimensions is given in Theorem 8.2.3 on page 247 of Conlon.

II. De Rham Cohomology

There is an obvious similarity between the condition $d_{q-1} \circ d_q = 0$ for the differentials in a singular chain complex and the condition $d[q] \circ d[q-1] = 0$ which is satisfied by the exterior derivative maps $d[k]$ on differential k-forms. The main difference is that the indices or gradings are reversed. In Section 1 we shall look more generally at graded sequences of algebraic objects $\{ A^k \}_{k \in \mathbf{Z}}$ which have mappings $\delta[k]$ from A^k to A^{k+1} such that the composite of two consecutive mappings in the family is always zero. This type of structure is called a **cochain complex**, and it is dual to a chain complex in the sense of category theory; every cochain complex determines *cohomology groups* which are dual to homology groups. We shall conclude Section 1 by explaining how every chain complex defines a family of cochain complexes. In particular, if we apply this to the chain complexes of smooth and continuous singular chains on a space (an open subset of $\mathbf{R}^n$ in the first case), then we obtain associated (smooth or continuous) *singular cohomology groups* for a space (with the previous restrictions in the smooth case) *with real coefficients* that are denoted by $S^*(X; \mathbf{R})$ and $S^*_{\mathrm{smooth}}(U; \mathbf{R})$ respectively. If U is an open subset of $\mathbf{R}^n$ then the natural chain maps $\varphi^{\#}$ from Section I.3 will define associated natural maps of chain complexes from continuous to smooth singular cochains that we shall call $\varphi_{\#}$, and there are also associated maps of the corresponding cohomology groups. In Section 2 we shall prove that the homology maps $\varphi^{\#}_*$ and cohomology maps $\varphi^*_{\#}$ are isomorphisms. This illustrates a phenomenon which already arose in 246A; namely, there are several different ways to define homology (and cohomology) groups, and each is particularly convenient in certain situations. In Section 3 we shall prove that De Rham cohomology has many of the basic formal properties that hold for singular cohomology. Finally, in Section 4 we prove an important result first discovered by G. De Rham in the 1930s: If U is an open subset of $\mathbf{R}^n$, then the generalized Stokes' Formula from Section I.3 defines a map J from the cochain complex $\wedge^*(U)$ of differential forms on U to the smooth singular cochain complex $S^*_{\mathrm{smooth}}(U; \mathbf{R})$, and **De Rham's Theorem** states that the associated map in cohomology J^* is an isomorphism. Some elementary consequences of this result will also be discussed.

II.1 : Smooth singular cochains

(Hatcher, § 2.1)

We begin by dualizing chain complexes and homology.

Definition. Let R be a commutative ring with unit. A *cochain complex* over R is a pair (C^*, δ^*) consisting of a sequence of R-modules C^q (the cochain modules) indexed by the integers, and coboundary homomorphisms $d^q : C^q \to C^{q+1}$ such that for all q we have $\delta^{q+1} \circ \delta^q = 0$. The *cocycles* in C^q are the elements x such that $\delta(x) = 0$ and the *coboundaries* in C^q are all elements x which are expressible as $\delta(y)$ for some y.

The defining conditions for a cochain complex imply that the image of δ^{q-1} is contained in the kernel of δ^q, and we define the q^{th} **cohomology module** $H^q(C)$ to be the quotient Kernel δ^q/Image δ^{q-1}.

Formally speaking, the notions of chain and cochain complex are categorically dual to each other. Given a chain complex (C_*, d_*), one can define the categorically dual cochain complex

(C^*, δ^*) by the equations $C^q = C_{-q}$ and $\delta^q = d_{1-q}$. Cochain complex morphisms can be defined by duality, and one has the following dualizations of standard results for chain complexes and their morphisms:

(1) *Algebraic morphisms of cochain complexes* $f : C \to D$ *pass to algebraic morphisms of cohomology groups* $[f] : H^*(C) \to H^*(D)$.

(2) *The algebraic morphisms in the preceding satisfy the conditions* $[g \circ f] = [g] \circ [f]$ *and* $[\mathrm{id}] = \mathrm{identity}$.

(3) *If we are given an exact sequence of cochain complexes* $0 \to A \to B \to C \to 0$ *(so that one has a short exact sequence* $0 \to A^q \to B^q \to C^q \to 0$*, then there is an associated long exact sequence of homology:*

$$\cdots \to H^{q-1}(C) \to H^q(A) \to H^q(B) \to H^q(C) \to H^{q+1}(A) \to \cdots$$

(4) *The long exact sequence in the previous statement is natural with respect to suitably defined morphisms of short exact sequences of cochain complexes.*

In each case, the proof is a straightforward dualization of the corresponding argument for chain complexes.

Cochain complexes associated to a chain complex

The main reason for introducing formal duals of chain complexes is that there are many situations in which it is necessary to work with both chain complexes and cochain complexes at the same time. The discussion of the Generalized Stokes' Formula in the preceding unit is one basic example. Our next step is to give a general method for constructing many different cochain complexes out of a chain complex.

Definition. Let (S_*, d_*) denote a chain complex over a commutative ring with unit A, and let M be an A-module (we assume all modules satisfy the identity $1 \cdot m = m$ for all m). The *complex of cochains on* S *with coefficient in* M is given by $C^q(S; M) = \mathrm{Hom}_A(S_q, M)$ (*i.e.*, the module of A-homomorphisms), and the coboundary map $\delta^q : C^q(S; M) \to C^{q+1}(S; M)$ is equal to the adjoint map $(d_{q+1})^*$ which takes a cochain (or function) $f : S_q \to M$ into $f \circ d_{q+1}$.

The identity $d_{q+2} \circ d_{q+1} = 0$ implies that $\delta^{q+1} \circ \delta^q = 0$, and therefore we do have a cochain complex

$$(\, C^*(S, M),\ \delta\,)$$

whose cohomology is called the *cohomology of* S *with coefficients in* M and written $H^*(S; M)$.

EXAMPLE. If X is a topological space and $S_*(X)$ is the singular chain complex of X, then for each abelian group M we obtain an associated singular cochain complex with coefficients in M, written $S^*(X; M)$. Likewise, if U is open in some $\mathbf{R}^n$ and $S_*^{\mathrm{smooth}}(U)$ is the smooth singular complex of U, then we have an associated smooth singular cochain complex with coefficients in M, written $S^*_{\mathrm{smooth}}(U; M)$. The Generalized Stokes' Formula implies that the integration map J defines a map of cochain complexes from $\wedge^*(U)$ to $S^*_{\mathrm{smooth}}(U; \mathbf{R})$.

One can now combine the previously described results on formal dualizations with the definitions of associated cochain complexes to obtain the following basic results:

PROPOSITION. *Suppose that $f : S \to T$ is a morphism of chain complexes over the ring A as above, and let M be an A-module as above. Then there are associated morphisms of cochain complexes*

$$f^{\#} : C^*(T; M) \longrightarrow C^*(S; M)$$

and morphisms of cohomology groups

$$f^* : H^*(T; M) \longrightarrow H^*(S; M)$$

which are **contravariantly** *functorial with respect to chain complex morphisms. Furthermore, if $g : M \to N$ is a homomorphism of A-modules, then there are associated morphisms of cochain complexes*

$$g_{\#} : C^*(S; M) \longrightarrow C^*(S; N)$$

and morphisms of cohomology groups

$$g_* : H^*(T; M) \longrightarrow H^*(T; N)$$

which are **covariantly** *functorial in with respect to module homomorphisms.*■

In particular, if we are given a continuous map of topological spaces $f : X \to Y$ and its associated map of singular chain complexes $f_{\#}$, then we obtain maps of singular cochain complexes $f^{\#} : S^*(Y; M) \to S^*(X; M)$ and morphisms of cohomology groups $f^* : H^*(Y; M) \to H^*(X; M)$ which are **contravariantly** functorial with respect to continuous mappings. Likewise, if we are given a smooth map of open subsets in Euclidean spaces $f : U \to V$ and its associated map of smooth singular chain complexes $f_{\#}$, then we obtain maps of singular cochain complexes $f^{\#} : S^*_{\mathrm{smooth}}(Y; M) \to S^*_{\mathrm{smooth}}(X; M)$ and morphisms of cohomology groups $f^* : H^*_{\mathrm{smooth}}(Y; M) \to H^*_{\mathrm{smooth}}(X; M)$ which are **contravariantly** functorial with respect to smooth mappings. Finally, for open subsets in Euclidean spaces the canonical natural transformation from $S_*^{\mathrm{smooth}}(U)$ to $S_*(U)$ defines natural transformations of cochain complexes

$$\varphi^{\#\#} : S^*(U; M) \longrightarrow S^*_{\mathrm{smooth}}(U; M)$$

and cohomology groups $H^*(U; M) \to H_{\mathrm{smooth}}(U; M)$ which are natural with respect to smooth maps.

As in the case of chain complexes, one immediate question is whether the smooth and ordinary definitions of singular chains for open subsets in $\mathbf{R}^n$ yield isomorphic groups. One aim of this unit is to develop enough machinery so that we can prove this, at least in some important special cases. The following question is clearly closely related:

PROBLEM. *Suppose that $f : S \to T$ is a map of chain complexes such that $f_* : H_*(S) \to H_*(T)$ is an isomorphism in homology. Under what conditions is $f^* : H^*(T; M) \to H^*(S; M)$ an isomorphism in cohomology?*

In this section we shall give some frequently occurring conditions under which the cohomology mappings are isomorphism.

We have not yet discussed versions of (3) and (4) for cochain complexes of the form $C^*(S;M)$ because there is a slight complication. If we have a short exact sequence of A-modules $0 \to S \to T \to U \to 0$, then it is fairly straightforward to show that the associated sequence of adjoint homomorphisms

$$0 \longrightarrow \mathrm{Hom}_A(U;M) \longrightarrow \mathrm{Hom}_A(T;M) \longrightarrow \mathrm{Hom}_A(S;M)$$

is exact, but the last map in this sequence is not always surjective. Simple examples can be constructed by taking the short exact sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_2 \to 0$ (in which the self map of $\mathbf{Z}$ is multiplication by 2) and setting M equal to either $\mathbf{Z}$ or $\mathbf{Z}_2$. **However,** if the short exact sequence is **split**, so that there is a map from U to T which yields a direct sum decomposition $T \cong S \oplus U$, then the associated sequence of adjoint homomorphisms will be exact, for the map from $\mathrm{Hom}_A(T;M)$ to $\mathrm{Hom}_A(T;M)$ will then be onto (verify this!). This leads directly to the following result.

PROPOSITION. *Suppose that we are given a short exact sequence of chain complexes $0 \to S \to T \to U \to 0$ such that for each q the short exact sequence $0 \to S_q \to T_q \to U_q$ splits (with no assumptions whether or not the maps $U_* \to T_*$ define a chain complex morphism). Then one has a short exact sequence of cochain complexes*

$$0 \longrightarrow C^*(U;M) \longrightarrow C^*(T;M) \longrightarrow C^*(S;M) \longrightarrow 0$$

and associated long exact sequences of cohomology. The latter are contravariantly functorial with respect to morphisms of long exact sequences of (suitably restricted) chain complexes, and they are covariantly functorial with respect to homomorphisms of the coefficient modules.■

This result applies directly to singular cochain complexes. If (X,A) is a pair of spaces with $A \subset X$, then the standard free generators of $S_*(A)$ have a natural interpretation as a subset of the standard free generators for $S_*(X)$, and therefore we have isomorphisms of chain groups $S_q(X) \cong S_q(A) \oplus S_*(X,A)$ for all q (however, such maps **rarely** define an isomorphism of chain complexes). A similar situation holds for smooth singular chains. Therefore, in both cases one has long exact cohomology sequences, and in fact there is a long commutative latter relating these two long exact sequences on the category of open subsets in Euclidean spaces.

The Kronecker index pairing

We have defined cochains to be objects that assign values to every chain, and we would like to have a similar principle in cohomology; namely, if C is a chain complex and M is a module, then a class in $H^q(C;M)$ assigns a value in M to every class in $H_q(M)$. This map turns out to be bilinear, and it is usually called the Kronecker index pairing.

Formally, proceed as follows: Given a cocycle u and a cycle z as above, define $\kappa(u,z) = \langle u,z\rangle \in M$ by choosing f representing u and c representing z and setting $\kappa(u,z) = f(c)$. We are immediately faced with proving the following result to show this is a valid definition.

LEMMA. *In the preceding discussion, if we are given other representatives $f + \delta\, g$ and $c + d(b)$ for u and z, then we obtain the same element in M.*

Sketch of proof. The value of $[f+gd](c+db)$ is given by

$$f(c) \;+\; gd(c) \;+\; fd(b) \;+\; gdd(b)\ .$$

In this expression the second term vanishes because $d(c)=0$, the third term vanishes because $fd=0$, and the final term vanishes because $dd=0$.■

The following result implies that the Kronecker index is often nontrivial:

PROPOSITION. *If A is a principal ideal domain and C is a chain complex of free A-modules, then the adjoint map $\kappa' : H^q(C;M) \to \mathrm{Hom}_A(H_q(C),M)$ defined by*

$$[\kappa'(u)](z) \quad = \quad \kappa(u,z) \quad \in \quad M$$

is onto.

Proof. Let $B_q \subset C_{q-1}$ denote the image of d_q, and let Z_q be the kernel of d_q. Then we have $C_q/Z_q \cong B_q$. Since we are working over a principal ideal domain, a submodule of a free module is free. Therefore we may define a one-sided inverse to the projection $C_q \to B_q$ by lifting a set of free generators in B_q to classes in C_q, and then taking the unique extension of this map to a homomorphism of A-modules. This immediately yields a direct sum decomposition

$$C_q \quad\cong\quad Z_q \,\oplus\, B_q$$

and the latter in turn implies that *every homomorphism from Z_q to a module M can be extended to C_q.*

How does this apply to prove the proposition? Suppose that we are given a homomorphism $\alpha : H_q(C) \to M$. Since the domain is the quotient module Z_q/B_{q+1}, it follows that we can pull α back to Z_q and obtain a homomorphism α_0 on the cycles. By the previous paragraph we can extend α_0 to a map α_1 on C_q; this map vanishes on B_{q+1} by construction, and this yields the cocycle condition $\delta(\alpha_1) = \alpha_1 \circ d_{q+1} = 0$. Therefore we conclude that $u = \kappa'([\alpha_1])$.■

There are simple examples to show that κ' is not always onto. Consider the chain complex given by $\mathbf{Z} \to \mathbf{Z}$, where the first copy of the integers is in degree 1 and the map is multiplication by $m > 1$. Then the only nontrivial homology group is $H_0 \cong \mathbf{Z}_m$, and direct computation also shows that the only nonzero cohomology group with integer coefficients is $H^1(C;\mathbf{Z}) \cong \mathbf{Z}_m$. In particular, since the group $\mathrm{Hom}(H_1,\mathbf{Z})$ is trivial, it follows that κ' cannot be injective in this case, and in fact it is the trivial homomorphism. However, the situation is better if we further specialize to chain complexes which are vector spaces over fields.

PROPOSITION. *In the setting of the lemma, if A is a field then κ' is an isomorphism.*

Proof. We already know that the map is onto, so it is only necessary to prove it is 1–1. Suppose now that $\kappa'(u) = 0$ and u is represented by the cocycle $f : C_q \to M$. Then we have $f(c) = 0$ for every cycle $c \in C_q$. This means that f factors into a composite

$$C_q \longrightarrow C_q/Z_q \;\cong\; B_q \longrightarrow M$$

and since we are working with vector spaces over a field we know that the map $B_q \to M$ extends to a homomorphism g on C_q. By construction we know that $f = \delta(g)$, and therefore we conclude that $u = 0$.■

There is a corresponding definition for homology with coefficients in an arbitrary A-module; for the time being we may assume A is an arbitrary commutative ring with unit. To simplify the discussion we shall assume that the chain complex (C_*, d_*) has chain groups C_q which are all free A-modules.

In terms of tensor products, the complex with coefficients is given by $C_* \otimes_A M$; computationally, this means that the elements of $C_q \otimes_A M$ have the form $\sum_i \gamma_i \otimes m_i$ where the γ_i lie in some fixed set of free generators for C_q and the m_i belong to M. The tensor product $a \otimes_A b$ has the standard bilinearity properties. There is an evident definition of mappings $d_q \otimes_A M$, and these make the sequence $C_q \otimes_A M$ into a chain complex, whose homology is called $H_q(C; M)$.

We recall that if P is an A-module and M is a commutative ring with unit such that the map $A \to M$ sending one unit to the other is a ring homomorphism, then $P \otimes_A M$ has a standard structure as an M-module.

In this section we are primarily interested in situations where A is the integers and M is a field; in this case the mappings in the chain complex turn out to be morphisms of vector spaces over fields and the homology groups have associated structures of vector spaces over the field M.

PROPOSITION. *In the setting described above, if M has characteristic zero (no sum of 1 with itself finitely many times yields zero), then there is a natural isomorphism of vector spaces from $H_q(C) \otimes M$ to $H_q(C; M)$.*

Proof. First of all, if $z \in H_q(C)$ and $m \in M$, then the mapping in question sends $z \otimes m$ to the class of $c \otimes m$, where c is a cycle representing z. One can check directly that this map is a well-defined and yields a morphism of vector spaces over M. The isomorphism statement follows because tensoring with the characteristic zero field M (in fact with an arbitrary group having no nonzero elements of finite order) sends short exact sequences into exact sequences. The latter means that the cycles in $C_* \otimes M$ are given by $Z_* \otimes M$, the boundaries in $C_* \otimes M$ are given by $B_* \otimes M$, and the homology is given by the quotient of these groups, which is just $H_* \otimes M$.■

COROLLARY. *If S and T are chain complexes of free abelian groups such that the chain map $f : S \to T$ defines an isomorphism in homology, then for each field M of characteristic zero the maps $f^* : H^*(T; M) \to H^*(S; M)$ are also isomorphisms.*

Proof. The tensor product construction takes isomorphisms to isomorphisms, so the map defined by f from $H_*(S) \otimes M$ to $H_*(T) \otimes M$ is an isomorphism. By the proposition it follows that $H_*(S; M)$ to $H_*(T; M)$ is also an isomorphism. Therefore the associated map of dual vector spaces is also an isomorphism. Since the Kronecker index pairing defines a natural isomorphism from the dual space of $H_*(C; M)$ to $H^*(C; M)$ for $C = S \otimes M$ or $T \otimes M$, it follows that these cohomology groups must also be isomorphic under the map defined by f.■

COROLLARY. *If U is an open subset in some Euclidean space, then the natural map from $H^*(U; \mathbf{R})$ to $H^*_{\mathrm{smooth}}(U; \mathbf{R})$ is an isomorphism.*

This is a special case of the preceding result.■

II.2 : Homological comparison theorem

(Hatcher, § 2.3)

The aim of this section is to show that the natural map from smooth singular chains to ordinary chains

$$S_*^{\mathrm{smooth}}(U) \longrightarrow S_*(U)$$

defines isomorphisms in homology and in cohomology with real coefficients if U is an arbitrary open subset of some $\mathbf{R}^n$.

It will be convenient to extend the definition of smooth singular chain complexes to arbitrary subsets of $\mathbf{R}^n$ for some n. Specifically, if $A \subset \mathbf{R}^n$ then the smooth singular chain complex $S_*^{\mathrm{smooth}}(A)$ is defined so that each group $S_q(A)$ is free abelian on the set of continuous mappings $T : \Lambda_q \to A$ which extend to smooth mappings T' from some open neighborhood $W(T')$ of Λ_q to $\mathbf{R}^n$. If A is an open subset of $\mathbf{R}^n$, then this is equivalent to the original definition, for if we are given T' as above we can always find an open neighborhood V of Λ_q such that T' maps V into A.

Clearly the definitions of smooth and ordinary singular chains are similar, and in fact many properties of ordinary singular chain complexes extend directly to smooth singular chain complexes. The following two are particularly important:

(0) If A is a convex subset of $\mathbf{R}^n$ (which is not necessarily open), then the constant map defines an isomorphism from $H_q^{\mathrm{smooth}}(A)$ to $H_q^{\mathrm{smooth}}(\mathbf{R}^0)$ for all q; in particular, these groups vanish unless $q = 0$.

(1) If we are given two smooth maps $f, g : U \to V$ such that f and g are smoothly homotopic, then the chain maps from $S_*^{\mathrm{smooth}}(U)$ to $S_*^{\mathrm{smooth}}(V)$ determined by f and g are chain homotopic.

(2) The construction of barycentric subdivision chain maps $\beta : S_*(U) \to S_*(U)$ in Section IV.4 of the 246A notes, and the related chain homotopy from β to the identity, determine compatible mappings of the same type on smooth singular chain complexes.

The first two of these follow because the chain homotopy constructions from Unit III of the 246A notes send smooth chains to smooth chains. The proof of the final assertion has two parts. First, the barycentric subdivision chain map in Section IV.4 of the 246A notes takes singular chains in the images of the canonical mappings

$$S_*^{\mathrm{smooth}}(W) \longrightarrow S_*(W)$$

into chains which also lie in the images of such mappings. However, the construction of the chain homotopy must be refined somewhat in order to ensure that it sends smooth chains to smooth chains. In order to construct such a refinement, one needs to know that the homology of $S_*^{\mathrm{smooth}}(\Lambda_q)$ is isomorphic to the homology of a point (hence is zero in positive dimensions). The latter is true by Property (0).■

As in the ordinary case, if $\mathcal{W}$ is an open covering of an open set $U \subset \mathbf{R}^n$, then one can define the complex $\mathcal{W}$-small singular chains

$$S_*^{\mathrm{smooth},\mathcal{W}}(U)$$

generated by all smooth singular simplices whose images lie inside a single element of $\mathcal{W}$, and the argument for ordinary singular chains implies that the inclusion map

$$S_*^{\mathrm{smooth},\mathcal{W}}(U) \longrightarrow S_*^{\mathcal{W}}(U)$$

defines isomorphisms in homology. The latter in turn implies that one has long exact Mayer-Vietoris sequences relating the smooth singular homology groups of U, V, $U \cap V$ and $U \cup V$, where U and V are open subsets of (the same) $\mathbf{R}^n$, and in fact one has a long commutative ladder diagram relating the Mayer-Vietoris sequences for (U, V) with smooth singular chains and ordinary singular chains.

The smooth and ordinary singular chain groups for $\mathbf{R}^0$ are identical, and therefore their smooth and ordinary singular homology groups are isomorphic under the canonical map from smooth to ordinary singular homology. By the discussion above, it follows that the canonical map

$$\varphi_*^U : S_*^{\mathrm{smooth}}(U) \longrightarrow S_*(U)$$

is an isomorphism if U is a convex open subset of some $\mathbf{R}^n$. The next step is to extend the class of open sets for which φ_*^U is an isomorphism.

THEOREM. *The map φ_*^U is an isomorphism if U is a finite union of convex open subsets in $\mathbf{R}^n$.*

Proof. Let (C_k) be the the statement that φ_*^U is an isomorphism if U is a union of at most k convex open subsets. Then we know that (C_1) is true. Assume that (C_k) is true; we need to show that the latter implies (C_{k+1}).

The preceding statements about ladder diagrams and the Five Lemma imply the following useful principle: *If we know that φ_*^U, φ_*^V, and $\varphi_*^{U \cap V}$ are isomorphisms in all dimensions, then the same is true for $\varphi_*^{U \cup V}$.* — Suppose now that we have a finite sequence of convex open subsets $W_1, \cdots, W_{k+1}$, and take U and V to be $W_1 \cup \cdots \cup W_k$ and W_{k+1} respectively. Then we know that φ_*^U and φ_*^V are isomorphisms by the inductive hypotheses. Also, since

$$U \cap V \quad = \quad (W_1 \cap W_{k+1}) \cup \cdots \cup (W_k \cap W_{k+1})$$

and all intersections $W_i \cap W_j$ are convex, it follows from the induction hypothesis that $\varphi_*^{U \cap V}$ is an isomorphism in all dimensions. Therefore by the observation at the beginning of this paragraph we know that $\varphi_*^{U \cup V}$ is an isomorphism, which is what we needed in order to complete the inductive step.■

To complete the proof that φ_*^U is an isomorphism for all U, we need the so-called *compact carrier properties* of singular homology. There are two versions of this result.

THEOREM. *Let X be a topological space, and let $u \in H_q(X)$. Then there is a compact subset $K \subset X$ such that u lies in the image of the canonical map from $H_q(K)$ to $H_q(X)$. Furthermore, if K is a compact subset of X, and v and w are classes in $H_q(K)$ whose images in $H_q(X)$ are equal, then there is a compact subset L such that $K \subset L \subset X$ such that the images of v and w are equal in $H_q(L)$.*

Proof. Choose a singular chain $\sum_i n_i T_i$ representing u, where each T_i is a continuous mapping $\Delta_q \to X$. If K is the union of the images $T_i[\Delta_q]$, then K is compact, and it follows that u lies in the image of $H_q(K)$ (because the chain lies in the subcomplex $S_*(K) \subset S_*(X)$.

To prove the second assertion in the proposition, note that by additivity it suffices to prove this when $w = 0$. Once again choose a representative singular chain $\sum_i n_i T_i$ for v; since the image of v in $H_q(X)$ is a boundary, there is a $(q+1)$-chain $\sum_j m_j U_j$ on X whose boundary is $\sum_i n_i T_i$. Let L be the union of K and the compact sets $U_j[\Delta_{q+1}]$; then L is compact and it follows immediately that v maps to zero in $H_q(L)$.■

We shall need a variant of the preceding result.

THEOREM. *Let U be an open subset of some $\mathbf{R}^n$, and let $u \in H_q^{\mathrm{CAT}}(U)$, where CAT denotes either ordinary singular homology or smooth singular homology. Then there is a finite union of convex open subsets $V \subset U$ such that u lies in the image of the canonical map from $H_q^{\mathrm{CAT}}(V)$ to $H_q^{\mathrm{CAT}}(U)$. Furthermore, if V is a finite union of convex open subsets of U, and v and w are classes in $H_q^{\mathrm{CAT}}(V)$ whose images in $H_q^{\mathrm{CAT}}(U)$ are equal, then there is a finite union of convex open subsets W such that $V \subset W \subset U$ such that the images of v and w are equal in $H_q^{\mathrm{CAT}}(W)$.*

Proof. The argument is similar, so we shall merely indicate the necessary changes. We adopt all the notation from the preceding discussion.

For the first assertion, by compactness we know that there is a finite union of convex open subsets V such that $K \subset V \subset U$, and it follows that u lies in the image of the homology of V. For the second assertion, take W to be the union of V and finitely many convex open subsets whose union contains L. It then follows that v maps to zero in the homology of W.■

We can now prove the following general result.

THEOREM. *The map φ_*^U is an isomorphism for arbitrary open subsets of some $\mathbf{R}^n$.*

Proof. If $u \in H_q(U)$, then we know there is some finite union of convex open subsets V such that $u = i_*(u_1)$, where $i : V \subset U$ is inclusion. By our previous results we know that $u_1 = \varphi_*^V(u_2)$ for some $u_2 \in H_q^{\mathrm{smooth}}(V)$, and since $i_* \circ \varphi_*^V = \varphi_*^U \circ i_*$, it follows that $u = \varphi_*^U i_*(u_2)$, so that φ_*^U is onto.

To show that φ_*^U is 1–1, suppose that v lies in its kernel. By the previous results we know that v lies in the image of $H_q^{\mathrm{smooth}}(V)$; suppose that v_1 maps to v. Then it follows that $v_2 = \varphi_*^V(v_1) \in H_q(V)$ maps to zero in $H_q(U)$, so that there is a finite union of convex open subsets W such that $V \subset W$ and v_2 maps to zero in $H_q(W)$. If $j : V \to W$ is inclusion, then it follows that $j_*(v_1)$ lies in the kernel of φ_*^W; however, we know that the latter map is 1–1 and therefore it follows that $j_*(v_1) = 0$. Since the image of the latter element in $H_*^{\mathrm{smooth}}(U)$ is equal to v, it follows that $v = 0$ and hence φ_*^U is 1–1, which is what we wanted to prove.■

If we combine this result with the observations in Section II.1, we immediately obtain a similar result for cohomology with real coefficients:

THEOREM. *If U is an arbitrary open subset of $\mathbf{R}^n$, then the map $\varphi_U^* : H^*(U;\mathbf{R}) \longrightarrow H^*_{\mathrm{smooth}}(U;\mathbf{R})$ is an isomorphism of real vector spaces.*■

II.3 : Eilenberg-Steenrod properties

(Hatcher, §§ 2.1, 2.3, 3.1; Conlon, § 2.6, 8.1, 8.3–8.5)

Definition. Let U be an open subset of $\mathbf{R}^n$ for some n. The **de Rham cohomology groups** $H^q_{\rm DR}(U)$ are the cohomology groups of the cochain complex of differential forms.

In Section 1 we noted that integration of differential forms defines a morphism J of chain complexes from $\wedge^*(U)$ to $S^*(U;\mathbf{R})$, where U is an arbitrary open subset of some Euclidean space. The aim of this section and the next is to show that the associated cohomology map $[J]$ defines an isomorphism from $H^*_{\rm DR}(U)$ to $H^*_{\rm smooth}(U;\mathbf{R})$; by the results of the preceding section, it will also follow that the de Rham cohomology groups are isomorphic to the ordinary singular cohomology groups $H^*(U;\mathbf{R})$. In order to prove that $[J]$ is an isomorphism, we need to show that the de Rham cohomology groups $H^*_{\rm DR}(U)$ satisfy analogs of certain formal properties that hold for (smooth) singular cohomology. One of these is a homotopy invariance principle, and the other is a Mayer-Vietoris sequence. Extremely detailed treatments of these results are given in Conlon, so at several points we shall be rather sketchy.

The following abstract result will be helpful in proving homotopy invariance. There are obvious analogs for other subcategories of topological spaces and continuous mappings, and also for covariant functors.

LEMMA. *Let T be a contravariant functor defined on the category of open subsets of $\mathbf{R}^n$ and smooth mappings. Then the following are equivalent:*

(1) *If f and g are smoothly homotopic mappings from U to V, then $T(f) = T(g)$.*

(2) *If U is an arbitrary open subset of $\mathbf{R}^n$ and $i_t : U \to U \times \mathbf{R}$ is the map sending u to (u,t), then $T(i_0) = T(i_1)$.*

Proof. (1) $\Longrightarrow$ (2). The mappings i_0 and i_1 are smoothly homotopic, and the inclusion map defines a homotopy from $U \times (-\varepsilon, 1+\varepsilon)$ to $U \times \mathbf{R}$.

(2) $\Longrightarrow$ (1). Suppose that we are given a smooth homotopy $H : U \times (-\varepsilon, 1+\varepsilon) \to V$. Standard results from 205C imply that we can assume the homotopy is "constant" on some sets of the form $(-\varepsilon, \eta) \times U$ and $(1-\eta, 1+\varepsilon) \times U$ for a suitably small positive number η. One can then use this property to extend H to a smooth map on $U \times \mathbf{R}$ that is "constant" on $(-\infty, \eta) \times U$ and $(1-\eta, \infty) \times U$. By the definition of a homotopy we have $H \circ i_1 = g$ and $H \circ i_0 = f$. If we apply the assumption in (1) we then obtain

$$T(g) \quad = \quad T(i_1) \circ T(H) \quad = \quad T(i_0) \circ T(H) \quad = \quad T(f)$$

which is what we wanted.■

A simple decomposition principle for differential forms on a cylindrical open set of the form $U \times \mathbf{R}$ will be useful. If U is open in $\mathbf{R}^n$ and I denotes the k-element sequence $i_1 < \cdots < i_k$, we shall write

$$\xi_I \quad = \quad dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and say that such a form is a *standard basic monomial k-forms* on U. Note that the wedge of two standard basic monomials $\xi_J \wedge \xi_I$ is either zero or ± 1 times a standard basic monomial, depending upon whether or not the sequences J and I have any common wedge factors.

PROPOSITION. *Every k-form on U is uniquely expressible as a sum*

$$\sum_I f_I(x,t)\, dt \wedge \xi_I \quad + \quad \sum_J g_J(x,t)\, \xi_J$$

where the index I runs over all sequences $0 < i_1 < \cdots < i_{k-1} \leq n$, the index J runs over all sequences $0 < j_1 < \cdots < j_k \leq n$, and f_I, g_J are smooth functions on $U \times \mathbf{R}$.■

We then have the following basic result.

THEOREM. *If U is an open subset of some $\mathbf{R}^n$ and $i_t : U \to U \times \mathbf{R}$ is the map $i_t(x) = (x,t)$, then the associated maps of differential forms $i_0^\#, i_1^\# : \wedge^*(U \times \mathbf{R}) \to \wedge^*(U)$ are chain homotopic.*

In this example the chain homotopy is frequently called a *parametrix.*

COROLLARY. *In the setting above the maps i_0^* and i_1^* from $H^*_{\mathrm{DR}}(U \times \mathbf{R})$ to $H^*_{\mathrm{DR}}(U)$ are equal.*■

Proof of Theorem. The mappings $P^q : \wedge^q(U \times \mathbf{R}) \to \wedge^{q-1}(U)$ are defined as follows. If we write a q-form over $U \times \mathbf{R}$ as a sum of terms $\alpha_I = f_I(x,t)\, dt \wedge \xi_I$ and $\beta_J = g_J(x,t)\, \xi_J$ using the lemma above, then we set $P^q(\beta_J) = 0$ and

$$P^q(\alpha_I) \quad = \quad \left(\int_0^1 f_I(x,u)\, du \right) \cdot \xi_I \ ;$$

we can then extend the definition to an arbitrary form, which is expressible as a sum of such terms, by additivity.

We must now compare the values of $dP + Pd$ and $i_1^\# - i_0^\#$ on the generating forms α_I and β_J described above. It follows immediately that $i_1^\#(\alpha_I) - i_0^\#(\alpha_I) = 0$ and

$$i_1^\#(\beta_J) \ - \ i_0^\#(\beta_J) \quad = \quad [g(x,1) \ - \ g(x,0)]\, \beta_J \ .$$

Next, we have $d \circ P(\beta_J) = d(0) = 0$ and

$$d \circ P(\alpha_I) \quad = \quad d \left(\int_0^1 f_I(x,u) du) \right) \cdot \xi_I \quad =$$

$$\sum_j \left(\int_0^1 \frac{\partial f_I}{\partial x^j}(x,u)\, du \right) \wedge dx^j \wedge \omega_I \ .$$

Similarly, we have

$$P \circ d(\alpha_I) \quad = \quad P \left(\sum_j \frac{\partial f_I}{\partial x^j}\, dx^j \wedge dt \wedge \xi_I \ + \ \frac{\partial f_I}{\partial t}\, dt \wedge dt \wedge \xi_I \right)$$

in which the final summand vanishes because $dt \wedge dt = 0$. If we apply the definition of P to the nontrivial summation on the right hand side of the displayed equation and use the identity $dx^j \wedge dt = -dt \wedge dx^j$, we see that the given expression is equal to $-d \circ P(\alpha_I)$; this shows that the values of both $dP + Pd$ and $i_1^\# - i_0^\#$ on α_I are zero. It remains to compute $P \circ d(\beta_J)$ and verify that it is equal to $i_1^\#(\beta_J) - i_0^\#(\beta_J)$. However, by definition we have

$$P \circ d(g_J\, \xi_J) \quad = \quad P\left(\sum_i \frac{\partial g_J}{\partial x^i}\, dx^i \wedge \xi_J \; + \; \frac{\partial g_J}{\partial t}\, dt \wedge \xi_J \right)$$

and in this case P maps the summation over i into zero because each form $dx^i \wedge \xi_J$ is either zero or ± 1 times a standard basic monomial, depending on whether or not dx^i appears as a factor of ξ_J. Thus the right hand side collapses to the final term and is given by

$$P\left(\frac{\partial g_J}{\partial t}\, dt \wedge \xi_J \right) \quad = \quad \left(\int_0^1 \frac{\partial g_J}{\partial u}(x,u)\, du \right) \xi_J \quad =$$

$$[\, g(x,1) \; - \; g(x,0)\,]\; \xi_J$$

which is equal to the formuula for $i_1^\#(\beta_J) - i_0^\#(\beta_J)$ which we described at the beginning of the argument.■

COROLLARY. *If U is a convex open subset of some $\mathbf{R}^n$, then $H^q_{\mathrm{DR}}(U)$ is isomorphic to $\mathbf{R}$ if $q = 0$ and is trivial otherwise.*

This follows because the constant map from U to $\mathbf{R}^0$ is a smooth homotopy equivalence if U is convex, so that the de Rham cohomology groups of U are isomorphic to the de Rham cohomology groups of $\mathbf{R}^0$, and by construction the latter are isomorphic to the groups described in the statement of the Corollary.■

COROLLARY. (Poincaré Lemma) *Let U be a convex open subset of some $\mathbf{R}^n$ and let $q > 0$. The a differential q-form ω on U is closed ($d\omega = 0$) if and only if it is exact ($\omega = d\theta$ for some θ).*■

Both of the preceding also hold if we merely assume that U is star-shaped with respect to some point $\mathbf{v}$ (*i.e.*, if $\mathbf{x} \in U$, then the closed line segment joining $\mathbf{x}$ and $\mathbf{v}$ is contained in U), for in this case the constant map is again a smooth homotopy equivalence.■

The Mayer-Vietoris sequence

Here is the main result:

THEOREM. *Let U and V be open subsets of $\mathbf{R}^n$. Then there is a long exact Mayer-Vietoris sequence in de Rham cohomology*

$$\cdots \to H^{q-1}_{\mathrm{DR}}(U \cap V) \to H^q_{\mathrm{DR}}(U \cup V) \to H^q_{\mathrm{DR}}(U) \oplus H^q_{\mathrm{DR}}(V) \to H^q_{\mathrm{DR}}(U \cap V) \to H^{q+1}_{\mathrm{DR}}(U \cup V) \to \cdots$$

and a commutative ladder diagram relating the long exact Mayer-Vietoris sequences for $\{U, V\}$ in de Rham cohomology and smooth singular cohomology with real coefficients.

Proof. The existence of the Mayer-Vietoris sequence will follow if we can show that there is a short exact sequence of chain complexes

$$0 \to \wedge^*(U \cup V) \longrightarrow \wedge^*(U) \oplus \wedge^*(V) \longrightarrow \wedge^*(U \cap V) \to 0$$

where the map from $\wedge^*(U \cup V)$ is given on the first factor by the $i_U^{\#}$ (where i_U denotes inclusion) and on the second factor by $-i_V^{\#}$, and the map into $\wedge^*(U \cap V)$ is given by the maps $j_U^{\#}$ and $j_V^{\#}$ defined by inclusion of $U \cap V$ into U and V.

The exactness of this sequence at all points except $\wedge^*(U \cap V)$ follows immediately. Therefore the only thing to prove is that the map to $\wedge^*(U \cap V)$ is surjective. This is done using smooth partitions of unity; details are given in Conlon (specifically, the last four lines of the proof for Lemma 8.5.1 on page 267).

The existence of the commutative ladder follows because the Generalized Stokes' Formula defines morphisms from the objects in the de Rham short exact sequence into the following analog for smooth singular cochains:

$$0 \to S^*_{\mathrm{smooth},\mathcal{U}}(U \cup V) \longrightarrow S^*_{\mathrm{smooth}}(U) \oplus S^*_{\mathrm{smooth}}(V) \longrightarrow S^*_{\mathrm{smooth}}(U \cap V) \to 0$$

The first term in this sequence denotes the cochains for the complex of $\mathcal{U}$-small chains on $U \cup V$, where $\mathcal{U}$ denotes the open covering $\{U, V\}$).

Since the displayed short exact sequence yields the long exact Mayer-Vietoris sequence for (smooth) singular cohomology, the statement about a commutative ladder in the theorem follows.■

II.4 : De Rham's Theorem

(Conlon, § 8.9)

The results of the preceding section show that the natural map $[J] : H^*_{\mathrm{DR}}(U) \to H^*_{\mathrm{smooth}}(U; \mathbf{R})$ is an isomorphism if U is a convex open subset of some Euclidean space, and if we compose this with the isomorphism between smooth and ordinary singular cohomology we obtain an isomorphism from the de Rham cohomology of U to the ordinary singular cohomology of U with real coefficients. The aim of this section is to show that both $[J]$ and its composite with the inverse map from smooth to ordinary cohomology is an isomorphism for an arbitrary open subset of $\mathbf{R}^n$. As in Section II.2, an important step in this argument is to prove the result for open sets which are expressible as finite unions of convex open subsets of $\mathbf{R}^n$.

PROPOSITION. *If U is an open subset of $\mathbf{R}^n$ which is expressible as a finite union of convex open subsets, then the natural map from $H^*_{\mathrm{DR}}(U)$ to $H^*_{\mathrm{smooth}}(U; \mathbf{R})$ and the associated natural map to $H^*(U; \mathbf{R})$ are isomorphisms.*

Proof. If W is an open subset in $\mathbf{R}^n$ we shall let ψ^W denote the natural map from de Rham to singular cohomology. If we combine the Mayer-Vietoris sequence of the preceding section with the considerations of Section II.2, we obtain the following important principle:

If $W = U \cup V$ and the mappings ψ^U, ψ^V and $\psi(U \cap V)$ are isomorphisms, then $\psi^{U \cup V}$ is also an isomorphism.

Since we know that ψ^V is an isomorphism if V is a convex open subset, we may prove the proposition by induction on the number of convex open subsets in the presentation $W = V_1 \cup \cdots \cup V_k$ using the same sorts of ideas employed in Section II.2 to prove a corresponding result for the map relating smooth and ordinary singular homology.■

The general case

Most open subsets of $\mathbf{R}^n$ are not expressible as finite unions of convex open subsets, so we still need some method for extracting the general case. The starting point is the following observation, which implies that an open set is a *locally finite* union of convex open subsets.

THEOREM. *If U is an open subset of $\mathbf{R}^n$, then U is a union of open subsets W_n indexed by the positive integers such that the following hold:*

(1) *Each W_n is a union of finitely many convex open subsets.*

(2) *If $|m - n| \geq 3$, then $W_n \cap W_m$ is empty.*

Proof. Results from 205C imply that U can be expressed as an increasing union of compact subsets K_n such that K_n is contained in the interior of K_{n+1} and K_1 has a nonempty interior. Define $A_n = K_n - \mathbf{Int}(K_{n-1})$, where K_{-1} is the empty set; it follows that A_n is compact. Let V_n be the open subset $\mathbf{Int}(K_{n+1}) - K_{n-1}$. By construction we know that V_n contains A_n and $V_n \cap V_m$ is empty if $|n - m| \geq 3$. Clearly there is an open covering of A_n by convex open subsets which are contained in V_n, and this open covering has a finite subcovering; the union of this finite family of convex open sets is the open set W_n that we want; by construction we have $A_n \subset W_n$, and since $U = \cup_n A_n$ we also have $U = \cup_n W_n$. Furthermore, since $W_n \subset V_n$, and $V_n \cap V_m$ is empty if $|n - m| \geq 3$, it follows that $W_n \cap W_m$ is also empty if $|n - m| \geq 3$.■

We shall also need the following result:

PROPOSITION. *Suppose that we are given an open subset U in $\mathbf{R}^n$ which is expressible as a countable union of pairwise disjoint subset U_k. If the map from de Rham cohomology to singular cohomology is an isomorphism for each U_k, then it is also an isomorphism for U.*

Proof. By construction the cochain and differential forms mappings determined by the inclusions $i_k : U_k \to U$ define morphisms from $\wedge^*(U)$ to the cartesian product $\Pi_k \wedge^*(U_k)$ and from $S^*_{\text{smooth}}(U)$ to $\Pi_k \; S^*_{\text{smooth}}(U_k)$. We claim that these maps are isomorphisms. In the case of differential forms, this follows because an indexed set of p-forms $\omega_k \in \wedge^p(U_k)$ determine a unique form on U (existence follows because the subsets are pairwise disjoint), and in the case of singular cochains it follows because every singular chain is uniquely expressible as a sum $\sum_k c_k$, where c_k is a singular chain on U_k and all but finitely many c_k's are zero (since the image of a singular simplex $T : \Delta_q \to U$ is pathwise connected and theopen sets U_k are pairwise disjoint, there is a unique m such that the image of T is contained in U_m).

If we are given an abstract family of cochain complexes C_k then it is straightforward to verify that there is a canonical homomorphism

$$H^* \left(\prod_k C_k \right) \longrightarrow \prod_k H^*(C_k)$$

defined by the projection maps

$$\pi_j : \prod_k C_k \longrightarrow C_j$$

and that this mapping is an isomorphism. Furthermore, it is natural with respect to families of cochain complex mappings $f_k : C_k \to E_k$.

The proposition follows by combining the observations in the preceding two paragraphs.■

We are now ready to prove the main result:

DE RHAM'S THEOREM. *The natural maps from de Rham cohomology to smooth and ordinary singular cohomology are isomorphisms for every open subset U in an arbitrary $\mathbf{R}^n$.*

Proof. Express U as a countable union of open subset W_n as in the discussion above, and for $k = 0, 1, 2$ let $U_k = \cup_m W_{3m+k}$. As noted in the definition of the open sets W_j, the open sets W_{3m+k} are pairwise disjoint. Therefore by the preceding proposition and the first result of this section we know that the natural maps from de Rham cohomology to singular cohomology are isomorphisms for the open sets U_k.

We next show that the natural map(s) must define isomorphisms for $U_1 \cup U_2$. By the highlighted statement in the proof of the first proposition in this section, it will suffice to show that the same holds for $U_1 \cap U_2$. However, the latter is the union of the pairwise disjoint open sets $W_{3m} \cap W_{3m+1}$, and each of the latter is a union of finitely many convex open subsets. Therefore by the preceding proposition and the first result of this section we know that the natural maps from de Rham to singular cohomology are isomorphisms for $U_1 \cap U_2$ and hence also for $U^* = U_1 \cup U_2$.

Clearly we would like to proceed similarly to show that we have isomorphisms from de Rham to singular cohomology for $U = U_0 \cup U^*$, and as before it will suffice to show that we have isomorphisms for $U_0 \cap U^*$. But $U_0 \cap U^* = (U_0 \cap U_1) \cup (U_0 \cap U_2)$, and by the preceding paragraph we know that the maps from de Rham to singular cohomology are isomorphisms for $U_0 \cap U_1$. The same considerations show that the corresponding maps are isomorphisms for $U_0 \cap U_2$, and therefore we have reduced the proof of de Rham's Theorem to checking that there are isomorphisms from de Rham to singular cohomology for the open set $U_0 \cap U_1 \cap U_2$. The latter is a union of open sets expressible as $W_i \cap W_j \cap W_k$ for suitable positive integers i, j, k which are distinct. The only way such an intersection can be nonempty is if the three integers i, j, k are *consecutive* (otherwise the distance between two of them is at least 3). Therefore, if we let

$$S_m \quad = \bigcup_{0 \leq k \leq 2} W_{3m-k} \cap W_{3m+1-k} \cap W_{3m+2-k}$$

it follows that S_m is a finite union of convex open sets, the union of the open sets S_m is equal to $U_0 \cap U_1 \cap U_2$, and if $m \neq p$ then $S_m \cap S_p$ is empty (since the first is contained in W_{3m} and the second is contained in the disjoint subset W_{3p}). By the first result of this section we know that the maps from de Rham to singular cohomology define isomorphisms for each of the open sets S_m, and it follows from the immediately preceding proposition that we have isomorphisms from de

Rham to singular cohomology for $\cup_m S_m = U_o \cap U_1 \cap U_2$. As noted before, this implies that the corresponding maps also define isomorphisms for U.■

Some examples

We shall now use de Rham's Theorem to prove a result which generalizes a theorem on page 551 of Marsden and Tromba's *Vector Calculus*:

THEOREM. *Suppose that $n \geq 3$ and $U \subset \mathbf{R}^3$ is the complement of some finite set X. If $\omega \in \wedge^1(U)$ is a closed 1-form, then $\omega = df$ for some smooth function f defined on U.*

Proof. It suffices to prove that $H^1_{\mathrm{DR}}(U) = 0$, and by de Rham's Theorem the latter is true if and only if $H^1(U;\mathbf{R})$ is trivial. If X consists of a single point, then U is homeomorphic to $S^{n-1} \times \mathbf{R}$ and the result follows because we know that $H_1(S^{n-1})$ and $H^1(S^{n-1};\mathbf{R})$ are trivial. We shall prove that $H^1(U;\mathbf{R})$ is trivial by induction on the number of elements in X.

Suppose that X has $k \geq 2$ elements and the result is known for finite sets with $(k-1)$ elements. Write $X = Y \cup \{z\}$ where $z \notin Y$, and consider the long exact Mayer-Vietoris sequence for $V = \mathbf{R}^n - Y$ and $W = \mathbf{R}^n - \{z\}$. Since $V \cup W = \mathbf{R}^n$ and $V \cap W = U$ we may write part of this sequence as follows:

$$H^1(V;\mathbf{R}) \oplus H^1(W;\mathbf{R}) \longrightarrow H^1(U;\mathbf{R}) \longrightarrow H^2(\mathbf{R}^n;\mathbf{R})$$

The induction hypothesis implies that the direct sum on the left is trivial, and the term on the right is trivial because $\mathbf{R}^n$ is contractible. This means that the term in the middle, which is the one we wanted to find, must also be trivial; the latter yields the inductive step in the proof.■

Generalization to arbitrary smooth manifolds

In fact, one can state and prove de Rham's Theorem for every (second countable) smooth manifold, and one approach to doing so appears in Conlon. We shall outline a somewhat different approach here and compare our approach with Conlon's.

The most fundamental point is that one can extend the definition of the differential forms cochain complexes to arbitrary smooth manifolds, and the associated functor for smooth mappings of open subsets of Euclidean spaces also extends to a contravariant functor on smooth maps of smooth manifolds. This is worked out explicitly in Conlon.

Next, we need to know that the cochain complexes of differential forms have the basic homotopy invariance properties described in the previous section. This is also shown in Conlon.

Given the preceding extensions, the generalization of de Rham's Theorem to arbitrary smooth manifolds reduces to the following basic fact:

THEOREM. *If M is an arbitrary second countable smooth n-manifold, then there is an open set U in some Euclidean space and a homotopy equivalence $f : M \to U$.*

This result is an immediate consequence of (i) the existence of a smooth embedding of M in some Euclidean space $\mathbf{R}^k$, which is shown in Subsection 3.7.C of Conlon, (ii) the Tubular Neighborhood Theorem for smooth embeddings, which is Theorem 3.7.18 in Conlon, (iii) the existence of a smooth homotopy equivalence from a manifold to its tubular neighborhood, which is discussed in Example 3.8.12 of Conlon.■

The application of all this to de Rham's Theorem is purely formal. By homotopy invariance the map f defines isomorphisms in de Rham and singular cohomology, and by the naturality of the integration map from de Rham to singular cohomology we have $\psi^M \circ f^* = f^* \circ \psi^U$. Since the maps f^* and ψ^U are isomorphisms, it follows that ψ^M must also be an isomorphism.■

COMPARISON WITH CONLON'S APPROACH. The approaches in Conlon and these notes boh require locally finite open coverings by subsets U_α such that each finite intersection $U_{alpha_1} \cap \cdots \cap U_{\alpha_k}$ is smoothly contractible. For open subsets in some Euclidean space, the existence of such coverings is an elementary observation; in the general case, one needs a considerable amount of differential geometry, including the existence of geodesically convex neighborhoods (Conlon, Section 10.5). Our approach eliminates the need to consider such neighborhoods or to work with Riemann metrics at all; instead, it uses results from Sections 3.7 and 3.8 of Conlon.

One important advantage of Conlon's approach is that it yields additional information about the isomorphisms in de Rham's Theorem. Namely, under these isomorphisms the wedge product on $H^*_{\mathrm{DR}}(M)$, given by the wedge product of differential forms, corresponds to the singular cup product on $H^*(M : \mathbf{R})$ as defined in Chapter 3 of Hatcher. Details are given in Section D.3 of Conlon.