Math 246B
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Homotopy classification of maps into S^1

One of the most basic problems in algebraic topology is to give an accessible algebraic description of the set $[X, Y]$ of homotopy classes from one "reasonable" space X to a second such space Y. The methods of algebraic topology yield an answer to this question if Y is the circle S^1 and X is a suitably restricted space, and in this document we shall prove half of the following result, which was originally published by N. K. Bruschlinsky in 1934:

Theorem. *Let X be a suitably restricted topological space, and let $x_0 \in X$ be an arbitrary basepoint. Let Δ be the map which takes the homotopy class of the function $f : X \to S^1$ to the homomorphism*

$$\widehat{f}_* : \pi_1(X, x_0) \to \pi_1(S^1, 1) \cong \mathbf{Z}$$

determined by the map $\widehat{f}(x) = f(x) \cdot f(x_0)^{-1}$, where the raised dot signifies ordinary complex multiplication on $S^1 \subset \mathbf{C}$. Then Δ defines a 1–1 correspondence from X to the abelian group

$$\mathrm{Hom}\,(\,\pi_1(X, x_0),\ \mathbf{Z}\,)$$

of algebraic homomorphisms from $\pi_1(X, x_0)$ to $\mathbf{Z}$.

More precisely, we shall prove the following:

Partial result. *Suppose that X is an arcwise connected, locally arcwise connected, Hausdorff space, and let Δ be as above. Then Δ defines a monomorphism onto a subgroup of the displayed abelian group* $\mathrm{Hom}\,(\,\pi_1(X, x_0),\ \mathbf{Z}\,)$.

If one places some additional "reasonable" conditions on the space X — for example, if one assumes that it admits a cell complex structure in the sense of 246A — then the map Δ is also onto. In general, the question of whether Δ is onto reduces to issues that are beyond the scope of this course (at least at this point).

PROOF OF PARTIAL RESULT. (*Sketch*) We can make $[X, S^1]$ into abelian group by pointwise multiplication of functions; clearly this can be done for representative functions, and it is elementary to prove that if f_0 is homotopy to f_1 and g_0 is homotopy to g_1, then their products $f_0 \cdot g_0$ and $f_1 \cdot g_1$ are homotopic. with this definition the mapping

$$\Delta : [X, S^1] \ \longrightarrow\ \mathrm{Hom}\,(\,\pi_1(X, x_0),\ \mathbf{Z}\,)$$

becomes a homomorphism, so it follows immediately that its image is a subgroup and that it is a monomorphism if $\Delta([f]) = 0$ implies that f is nullhomotopic.

Suppose then that $\Delta([f]) = 0$. This means that the basepoint preserving mapping $\widehat{f}$ lifts to a map $g : X \to \mathbf{R}$; in other words we have $p \circ g = \widehat{f}$, where $p(t) = \exp(2\pi\, i\, t)$. Since $\mathbf{R}$ is contractible it follows that g — and hence $\widehat{f}$ — must be nullhomotopic.

Finally, let γ be a continuous curve in S^1 such that $\gamma(0) = f(x_0)$ and $\gamma(1) = 1$ (how can one construct such a curve?). Then $H(x, t) = f(x) \cdot \gamma^{-1}(t)$ defines a homotopy from $\widehat{f}$ to f, and therefore f must be nullhomotopic.∎

Exercise. Fill in the details for all the unproved assertions in the preceding argument.