Preface

Formally, this course is a continuation of Mathematics 246A. However, it is not necessary to have taken the latter in order to start with this course for several reasons. First of all, we shall begin by taking a different approach to algebraic topology which does not immediately require knowledge of the constructions from 246A. There will be clear indications when we need input from that course, and references will be given. The somewhat different approach may shed some light on the abstract constructions that were necessary in 246A, and with hindsight the reasons for some of them may become more apparent.

We shall begin with a review *differential forms*, which are ordinarily covered in Mathematics 205C. One goal of the course is to give a topological explanation of the difference between closed and exact differential forms. Everything will start slowly, and at the beginning we shall consider the special case of interpreting the difference between the two types of differential forms in terms of the fundamental group. There is considerable overlap between these results and the Cauchy-Goursat Theorem in the theory of functions of a complex variable.

One reason for beginning slowly is to provide some opportunity for simultaneous review of some topics in 246A. More will be said about this in the main part of the course notes, but here is a start.

REVIEW SUGGESTION. This might be a good time to review the Preface from the 246A notes. The latter are available online at the following source:

`http://math.ucr.edu/∼res/math246A/algtopnotes.pdf`

This is probably also a good point to give the reference for the official course text.

> **A. Hatcher.** *Algebraic Topology* (Third Paperback Printing), *Cambridge University Press, New York NY*, 2002. ISBN: 0–521–79540–0.

This book can be legally downloaded from the Internet at no cost for personal use, and here is the link to the online version:

`www.math.cornell.edu/∼hatcher/AT/ATpage.html`

At some points we shall follow the text closely, but at others we shall go in different directions. The latter applies particularly to our discussion of algebraic topology and differential forms. One background reference for the latter is the following textbook for 205C:

> **L. Conlon.** Differentiable Manifolds. (Second Edition), *Birkhäuser-Boston, Boston MA*, 2001, ISBN 0–8176–4134–3.

Throughout the course we shall also use the following textbook for 205A and 205B as a reference for many topics and definitions:

J. R. Munkres. *Topology* (Second Edition), *Prentice-Hall, Saddle River NJ*, 2000. ISBN: 0–13–181629–2.

Content and objectives of this course

One of the central ideas in 246A was the definition of abelian groups called *homology groups* whose algebraic structure reflects many of the topological properties of a space. In particular, one can roughly think of the elements of homology groups as suitably defined equivalence classes of "nice" subspaces like closed curves or closed surfaces or, more generally, compact unbounded k-manifolds for suitable choices of k. An equally central idea in this course is the definition of **cohomology groups**, where as usual in mathematics the prefix **co-** indicates some sort of dual construction (say like the construction of a dual space associated to a vector space over a field). It is often convenient to view elements of cohomology groups as suitably defined equivalence classes of *measurement data* on the "nice" subspaces which represent elements of homology groups.

EXAMPLE. Suppose that $\omega = P\,dx + Q\,dy$ is a differential 1-form on an open connected subset U of $\mathbf{R}^2$ which is *closed* in the sense that its partial derivatives satisfy $P_y = Q_x$. Elements of the 1-dimensional homology of U are given by suitably defined equivalence classes of closed curves (specifically, one takes the unique maximal abelian quotient group of $\pi_1(U, u)$ for some arbitrary basepoint u). Given a smooth curve Γ we know how to define the line integral $\int_\Gamma \omega$, and it turns out that one can extend this definition to nonsmooth curves in a halfway reasonable manner. This line integral can then be viewed as defining measurement data for all closed curves, and it determines a 1-dimensional cohomology class of U. This class turns out to be zero if ω is an **exact** differential 1-form, but otherwise it might be nonzero.

In the Preface to the 246A notes we gave a few examples to indicate how the material from that course could shed some light on questions of independent interest. Here is a corresponding discussion for the present course:

1. One primary goal will be to give a unified approach to certain results in multivariable calculus involving the ∇ operator, Green's Theorem, Stokes' Theorem and the Divergence Theorem, and to formulate analogs for higher dimensions.

2. A related goal will be to provide a topological interpretation of certain consequences of the theorems mentioned in the preceding sentence. For example, if U is an open subset of $\mathbf{R}^2$ and ω is a closed 1-form on U as described above, then experience shows that there are often not too many possibilities for the value of the line integral $\int_\Gamma \omega$ where Γ ranges over all closed piecewise smooth curves in U. In particular, if U is the complement of a finite set of points, the results of this course will show that there are only countably many possible values for such line integrals.

3. Here is a question in a much different direction. On the torus $T^2 = S^1 \times S^1$ there are two standard closed curves given by the images of $\{1\} \times S^1$ and $S^1 \times \{1\}$. Of course, these curves meet in a single point, and one might ask if it is possible to find closed curves γ_1 and γ_2 which are (non-basepoint-preservingly) homotopic to these curves such that the images of γ_1 and γ_2 are disjoint. Experimentation suggests this is not possible, and the results obtained in this course will yield a mathematical proof that one can never find such curves.

4. In the preceding course we mentioned that a major goal of algebraic topology is to provide a setting for analyzing the set of homotopy classes of mappings $[X, Y]$ from one "nice space" X to a second "nice" space Y. The methods of Mathematics 205B and 246A yield a positive result in one simple but important case; specifically, if $Y = S^1$ and X is suitably restricted, then there is a canonical monomorphism from $[X, S^1]$ to the abelian group

$$\mathrm{Hom}\,(\,\pi_1(X, x_0),\ \mathbf{Z})$$

of algebraic homomorphisms from $\pi_1(X, x_0)$ to the infinite cyclic group $\mathbf{Z}$ (where x_0 is a suitably chosen basepoint), and it is given by taking a map $f : X \to S^1$ to the algebraic homomorphism from $\pi_1(X, x_0)$ to $\pi_1(S^1, 1) \cong \mathbf{Z}$ determined by the basepoint preserving map $f \cdot f(x_0)^{-1}$, where the raised dot indicates the usual complex multiplication on $S^1 \subset \mathbf{C}$. Furthermore, if X has a finite cell complex structure in the sense of 246A, then this monomorphism is an isomorphism (see the online document `bruschlinsky.pdf` for details and an alternate formulation in terms of results from 246A). One goal of the present course is to give a partial analog of these results if one replaces S^1 by S^n.

5. A more general issue involves the applicability of concepts from category theory. Algebraic topology is one branch of mathematics in which such ideas have been used successfully to solve questions of independent interest and to discover new and important phenomena. One objective of the course is to describe a few of these discoveries.

The keyed outline for the course (`math246Bkeyedoutline.pdf`) is very ambitious, and it is likely that not everything in the final units of the outline outline can be covered. However, priority will be given to two topics with ties to 246A; namely, the Alexander Duality Theorem — which places the classical Jordan-Brouwer separation theorems into a more general setting — and the Lefschetz Fixed Point Theorem — which may be viewed as a generalization of the Brouwer Fixed Point Theorem to spaces that are not contractible.

Background and review

This is a good time to read the section of the 246A notes called "Prerequisites." In this course we shall also use material from 205C when necessary, but most of the material will involve differential forms on open subsets of $\mathbf{R}^n$ for some n. A revised and slightly corrected version of a handout for this topic from Mathematics 205C

`extforms2007.ps`

is available in the course directory; the third section can be skipped on first reading because it only plays a limited role in the present course. Also, the following multivariable calculus textbook may be useful in connection with some examples we shall discuss:

> **J. E. Marsden and A. J. Tromba.** *Vector Calculus* (Fifth Edition), *W. H. Freeman & Co., New York NY*, 2003. ISBN: 0–7147–4992–0.

Portions of this course deal with the interaction between algebraic topology and other branches of mathematics, so it will be necessary to make some compromises in order to cover everything. Since the main emphasis of the course is on algebraic topology, we shall do this by varying the level of coverage; specifically, we shall try to make the topological content self-contained (at least when combined with the basic course references), but when we are considering interactions with other parts of mathematics, we shall sometimes assume whatever is needed from such areas.