I. Differential Forms and their Integrals

The purpose of this unit is to continue the discussion of differential forms from 205C in several directions. One objective is to prove a version of the result stating that the line integral of a closed 1-form over a closed curve only depends upon the homotopy class of the curve in the open set U on which the 1-form and curve are defined. A second objective is to relate the definition of integrals from 205C to the sorts of constructions one sees in multivariable calculus courses, and the third objective is to give a generalization of classical results in vector analysis (like Stokes' Theorem) to arbitrary dimensions.

I.1 : Differential 1–forms and the fundamental group

(Conlon, §§ 6.2 – 6.4)

In multivariable calculus one learns that certain line integrals in the plane of the form

$$\int_\Gamma P\,dx + Q\,dy$$

depend only on the endpoints of Γ. More precisely, if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

and P and Q have continuous partial derivatives on a **convex** open set, then the integral does not depend upon the path. In contrast, if we consider the line integral

$$\int_\Gamma \frac{x\,dy - y\,dx}{x^2+y^2}$$

over the counterclockwise unit circle $(\cos t, \sin t)$ for $0 \le t \le 2\pi$, then direct computation shows that the value obtained is 2π, but if we consider the corresponding line integral over the counterclockwise circle of radius $\frac{1}{3}$ centered at $(\frac{2}{3}, 0)$ with parametrization

$$x(t) = \tfrac{2}{3} + \tfrac{1}{3}\cos t, \qquad y(t) = \tfrac{1}{3}\sin t \qquad (0 \le t \le 2\pi)$$

then direct computation shows that the integral's value is zero. Since both of the curves we have described start and end at $(1,0)$, obviously the line integral does depend upon the path in this case. It is natural to ask the extent to which the line integral does vary with the choice of path; the main result here states that the value depends only on the homotopy class of the path, where it is assumed that the homotopy keeps the endpoints fixed.

MAIN RESULT. *Let U be an open subset of the coordinate plane, let $P(x,y)$ and $Q(x,y)$ be two functions with continuous partials satisfying the previous condition on partial derivatives, and* *let Γ and Γ' be two piecewise smooth curves in U with the same endpoints such that Γ and Γ' are homotopic by an endpoint preserving homotopy. Then*

$$\int_{\Gamma} P\,dx + Q\,dy = \int_{\Gamma'} P\,dx + Q\,dy\ .$$

In particular, if Γ and Γ' are closed curves, then the line integrals agree if Γ and Γ' determine the same element of $\pi_1(U, \{\text{endpoint}\})$.

At the end of this section we shall explain how this result yields a complete description of all values that the line integral

$$\int_{\Gamma} P\,dx + Q\,dy = \int_{\Gamma'} P\,dx + Q\,dy$$

can take, where P and Q are the specific functions above and Γ is a closed piecewise smooth curve whose image lies in $\mathbf{R}^2 - \{(0,0)\}$.

Relation to the Cauchy-Goursat Theorem. If f is an analytic function of a complex variable on the open set $U \subset \mathbf{R}^2 = \mathbf{C}$ and we write $f = u + i\,v$ as usual, then for every piecewise smooth curve Γ we have

$$\int_{\Gamma} f(z)\,dz \quad = \quad \int_{\Gamma} u\,dx \ - \ v\,dy \ + \ i \cdot \int_{\Gamma} v\,dx \ + \ u\,dy$$

and by the Cauchy-Riemann Equations the integrands of the two summands satisfy the previously formulated condition $P_y = Q_x$, and hence if u and v are known to have continuous partial derivatives then the main result proves a reasonably good form of the Cauchy-Goursat Theorem. Since the usual definition of analytic function does not include the continuity assumption for the partial derivatives, the main result does not quite prove the entire Cauchy-Goursat Theorem, but it is possible to modify the argument slightly in order to obtain the general result in which one does not assume the partial derivatives are continuous (if one continues to develop the subject of complex variables, it turns out that u and v always have continuous partial derivatives, but this requires additional work).

The following alternate version of the Main Theorem is frequently found in books on multivariable calculus.

ALTERNATE STATEMENT OF MAIN THEOREM. *If P and Q are as above, U is a connected region, and Γ is a piecewise smooth closed curve that is homotopic to a constant in U, then*

$$\int_{\Gamma} P\,dx + Q\,dy \quad = \quad 0\ .$$

This follows immediately from the Main Result, observation (3) above, and the triviality of the fundamental group of U. Conversely, the Main Result follows from the Alternate Statement. To see this, in the setting of the Main Result the curve $\Gamma' + (-\Gamma)$ is a closed piecewise smooth curve that is homotopic to a constant (verify this!), so the Alternate Statement implies that the line integral over this curve is zero. On the other hand, this line integral is also the difference of the line integrals over Γ' and Γ. Combining these observations, we see that the line integrals over Γ' and Γ must be equal.

In fact, it will be more convenient for us to prove the Alternate Statement in the discussion below.

The next result is often also found in multivariable calculus texts.

COROLLARY. *If in the setting of the Main Result and its Alternate Statement we also know that the region U is simply connected then*

(*i*) *for every piecewise smooth closed curve Γ in U we have*

$$\int_{\Gamma} P\,dx \; + \; Q\,dy \quad = \quad 0$$

(*ii*) *for every pair of piecewise smooth curves Γ, Γ' with the same endpoints we have*

$$\int_{\Gamma} P\,dx \; + \; Q\,dy \quad = \quad \int_{\Gamma'} P\,dx \; + \; Q\,dy \; .$$

The first part of the corollary follows from the triviality of the fundamental group of U, the Alternate Statement of the Main Result, and the triviality of line integrals over constant curve. The second part follows formally from the first in the same way that the Main Result follows from its Alternate Statement.

Background from multivariable calculus

As noted above, the following result can be found in most multivariable calculus textbooks.

PATH INDEPENDENCE THEOREM. *Let U be a rectangular open subset of the coordinate plane of the form $(a_1, b_1) \times (a_2, b_2)$, let P and Q be functions with continuous partials on U such that*

$$\frac{\partial P}{\partial y} \quad = \quad \frac{\partial Q}{\partial x}$$

and let Γ and Γ' be two piecewise smooth curves in U with the same endpoints. Then

$$\int_{\Gamma} P\,dx + Q\,dy \quad = \quad \int_{\Gamma'} P\,dx + Q\,dy \; .$$

The underlying idea behind the proof is to construct a function f such that $\nabla f = (P, Q)$. Green's Theorem plays major role in showing that the partials of f have the desired values.

Notational and abuse of language conventions. Given two points $\mathbf{p} = (p_1, p_2)$ and $\mathbf{q} = (q_1, q_2)$ in the coordinate plane, the closed straight line segment joining them is the curve $[\mathbf{p}, \mathbf{q}]$ with parametrization

$$x(t) \quad = \quad tp_1 + (1-t)p_2, \qquad y(t) \quad = \quad tq_1 + (1-t)q_2 \qquad (0 \leq t \leq 1) \; .$$

We would also like to discuss broken line curves, say joining $\mathbf{p}_0$ to $\mathbf{p}_1$ by a straight line segment, then joining $\mathbf{p}_1$ to $\mathbf{p}_2$ by a straight line segment, and so on. The points $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, etc. are called the *vertices* of the broken line curve. One technical problem with this involves the choices of linear parametrizations for the pieces. However, since line integrals for such curves do not depend upon such parametrizations and in fact we have

$$\int_C P\,dx + Q\,dy \quad = \quad \sum_i \int_{[\mathbf{p}_{i-1},\mathbf{p}_i]} P\,dx + Q\,dy$$

we shall not worry about the specific choice of parametrization. Filling in the details will be left as an exercise to a reader who is interested in doing so; this is basically elementary but tedious.

Integrals over broken line inscriptions

First some standard definitions. A *partition* of the interval $[a,b]$ is a sequence of points

$$\Delta : a \quad = \quad t_0 \quad < \quad t_1 \quad < \quad \cdots \quad < \quad t_m \quad = \quad b$$

and the *mesh* of Δ, written $|\Delta|$, is the maximum of the differences $t_i - t_{i-1}$ for $1 \leq i \leq m$. Given a piecewise smooth curve Γ defined on $[a,b]$, the *broken line inscription* $\mathbf{Lin}\,(\Gamma,\Delta)$ is the broken line curve with vertices

$$\Gamma(a) \quad = \quad \Gamma(t_0), \Gamma(t_1), \cdots \Gamma(t_m) \quad = \quad \Gamma(b)\ .$$

We are now ready to prove one of the key technical steps of the proof of the main result.

LEMMA. *Let U, be a connected open subset of $\mathbf{R}^2$, and let P, Q and Γ be as usual, where Γ is defined on $[a,b]$ and P and Q satisfy the condition*

$$\frac{\partial P}{\partial y} \quad = \quad \frac{\partial Q}{\partial x}\ .$$

Then there is a positive constant $\delta > 0$ such that for all partitions Δ of $[a,b]$ with $|\Delta| < \delta$ we have

$$\int_\Gamma P\,dx + Q\,dy \quad = \quad \int_{\mathbf{Lin}\,(\Gamma,\Delta)} P\,dx + Q\,dy\ .$$

Proof. If K is the image of Γ then K is a compact subset of the open set U, and therefore there is an $\varepsilon > 0$ so that if $\mathbf{x} \in \mathbf{R}^2$ satisfies $|\mathbf{x} - \mathbf{v}| < \varepsilon$ for some $\mathbf{v} \in K$ then $\mathbf{x} \in U$. It follows that if $\mathbf{v} \in K$ then the inner region for the square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon\sqrt{2}$ lies entirely in U.

By uniform continuity there is a $\delta > 0$ so that if $s, t \in [a,b]$ satisfy $|s-t| < \delta$ then

$$|\Gamma(s) - \Gamma(t)| \quad < \quad \frac{\varepsilon\sqrt{2}}{2}\ .$$

Let Δ be a partition of $[a,b]$ whose mesh is less than δ. Then for all i the restriction of Γ to $[t_{i-1}, t_i]$ lies in the open disk of radius $\frac{1}{2}\varepsilon\sqrt{2}$. It follows that both this restriction and the closed straight line segment joining $\Gamma(t_{i-1})$ to $\Gamma(t_i)$ lie in the open square region centered at $\Gamma(t_{i-1})$ with sides parallel to the coordinate axes of length of length $\varepsilon\sqrt{2}$; since the latter lies entirely in U. it follows that P and Q are defined on this square region. Therefore, by the previously quoted result from multivariable calculus we have

$$\int_{\Gamma|[t_{i-1},t_i]} P\,dx + Q\,dy \quad = \quad \int_{[\Gamma(t_{i-1}),\Gamma(t_i)]} P\,dx + Q\,dy$$

for each i. But the line integral over Γ is the sum of the line integrals over the curves $\Gamma|[t_{i-1},t_i]$, and the line integral over the broken line inscription is the sum of the line integrals over the line segments $[\Gamma(t_{i-1}),\Gamma(t_i)]$, and therefore it follows that the line integral over Γ is equal to the line integral over the broken line inscription, as required.

Proof of the Alternate Statement of the Main Result

We may as well assume that Γ is defined on the unit interval $[0,1]$ since we can always arrange this by a linear change of variables. Let $H : [0,1]\times[0,1] \to U$ be a continuous map such that $H(s,0) \;=\; \Gamma(s)$ for all s and H is constant on both $[0,1]\times\{1\}$ and $\{0,1\}\times[0,1]$.

If L is the image of H then L is a compact subset of the open set U, and as in the proof of the lemma there is an $\varepsilon' > 0$ so that if $\mathbf{x} \in \mathbf{R}^2$ satisfies $|\mathbf{x}-\mathbf{v}| < \varepsilon'$ for some $\mathbf{v} \in L$ then $\mathbf{x} \in U$. It follows that if $\mathbf{v} \in L$ then the inner region for the square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon'\sqrt{2}$ lies entirely in U.

By uniform continuity there is a $\delta' > 0$ so that if $\mathbf{s},\mathbf{t} \in [0,1]\times[0,1]$ satisfy $|\mathbf{s}-\mathbf{t}| < \delta'$ then

$$|H(\mathbf{s}) - H(\mathbf{t})| \quad < \quad \frac{\varepsilon'\sqrt{2}}{2}\ .$$

Without loss of generality we may assume that δ' is no greater than the δ in the previous lemma. Let Δ be a partition of $[a,b]$ whose mesh is less than $\frac{1}{2}\delta'\sqrt{2}$, and choose a positive integer N such that

$$\frac{1}{N} \quad < \quad \frac{\delta'\sqrt{2}}{2}\ .$$

Then for all i such that $1 \leq i \leq m$ and all j such that $1 \leq j \leq N$ the restriction of H to $[t_{i-1},t_i]\times\left[\frac{j-1}{N},\frac{j}{N}\right]$ lies in an open disk of radius $\frac{1}{2}\varepsilon'\sqrt{2}$.

A special case. To motivate the remainder of the argument, we shall first specialize to the case where H extends to a map on an open set containing the square $[0,1]\times[0,1]$ and has continuous partials on this open set. For each i such that $0 \leq i \leq m$ and each j such that $1 \leq j \leq N$ let $A(i,j)$ be the broken line curve in the square with vertices

$$\left(0,\tfrac{j-1}{N}\right),\ \ldots\ \left(t_i,\tfrac{j-1}{N}\right),\left(t_i,\tfrac{j}{N}\right),\ \ldots\ \left(1,\tfrac{j}{N}\right)\ .$$

In other words, this curve is formed by starting with a horizontal line segment from $\left(0,\frac{j-1}{N}\right)$ to $\left(t_i,\frac{j-1}{N}\right)$, then concatenating with a vertical line segment from $\left(t_i,\frac{j-1}{N}\right)$ to $\left(t_i,\frac{j}{N}\right)$, and finally concatenating with a horizontal line segment from $\left(t_i,\frac{j}{N}\right)$ to $\left(1,\frac{j}{N}\right)$. If $W(i,j)$ denotes the composite $H\circ A(i,j)$, then it follows that $W(i,j)$ is a piecewise smooth closed curve in U. Furthermore, $W(m,1)$ is just the concatenation of Γ with a constant curve and $W(0,N)$ is just a constant curve, so the proof of the main result reduces to showing that the line integrals of the expression

$P\,dx + Q\,dy$ over the curves $W(m,1)$ and $W(0,N)$ are equal. We claim this will be established if we can show the following hold for all i and j:

(1) The corresponding line integrals over the curves $W(0, j-1)$ and $W(m,j)$ are equal.

(2) The corresponding line integrals over the curves $W(i-1, j)$ and $W(i,j)$ are equal.

To prove the claim, first note that (2) implies that the value of the line integral over $W(i,j)$ is a constant z_j that depends only on j, and then note that (1) implies $z_{j-1} = z_j$ for all j. Thus the two assertions combine to show that the line integrals over all the curves $W(i,j)$ have the same value.

We begin by verifying (1). Since H is constant on $\{0,1\} \times [0,1]$, it follows that $W(m,j)$ is formed by concatenating $H|[0,1] \times \{\frac{j}{m}\}$ and a constant curve (in that order), while $W(0, j-1)$ is formed by concatenating a constant curve and $H|[0,1] \times \{\frac{j}{m}\}$ (again in the given order). Thus the line integrals over both $W(0, j-1)$ and $W(m,j)$ are equal to the line integral over $H|[0,1] \times \{\frac{j}{m}\}$, proving (1).

Turning to (2), since the broken line curves $A(i,j)$ and $A(i-1,j)$ differ only by one vertex, it follows that the difference

$$\int_{W(i,j)} P\,dx \ + \ Q\,dy \ - \ \int_{W(i-1,j)} P\,dx \ + \ Q\,dy$$

is equal to

$$\int_{V(i,j)} P\,dx \ + \ Q\,dy \ - \ \int_{V'(i,j)} P\,dx \ + \ Q\,dy$$

where $V(i,j)$ is the composite of H with the broken line curve with vertices

$$\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad \left(t_i, \tfrac{j-1}{N}\right), \quad \left(t_i, \tfrac{j}{N}\right)$$

and $V'(i,j)$ is the composite of H with the broken line curve with vertices

$$\left(t_{i-1}, \tfrac{j-1}{N}\right), \quad \left(t_{i-1}, \tfrac{j}{N}\right), \quad \left(t_i, \tfrac{j}{N}\right) \ .$$

Our hypotheses imply that both of these curves lie in an open disk of radius $\frac{1}{2}\varepsilon'\sqrt{2}$ and thus also in the open square centered at $\mathbf{v}$ with sides parallel to the coordinate axes of length $\varepsilon'\sqrt{2}$; by construction the latter retion lies entirely in U. Therefore by the previously quoted result from multivariable calculus we have

$$\int_{V(i,j)} P\,dx + Q\,dy \quad = \quad \int_{V'(i,j)} P\,dx + Q\,dy$$

for each i and j, so that the difference of the line integrals vanishes. Since this difference is also the difference between the line integrals over $W(i,j)$ and $W(i-1,j)$, it follows that the line integrals over the latter two curves must be equal.

The general case. If H is an arbitrary continuous function the preceding proof breaks down because we do not know if the continuous curves $W(i,j)$ are well enough behaved to define line integrals. We shall circumvent this by using broken line approximations to these curves and appealing to the previous lemma to relate the value of the line integrals over these approximations to the value on the original curve. Since the proof is formally analogous to that for the special case we shall concentrate on the changes that are required.

Let $X(i,j)$ denote the broken line curve with vertices

$$H\left(0,\tfrac{j-1}{N}\right), \ldots H\left(t_i,\tfrac{j-1}{N}\right), H\left(t_i,\tfrac{j}{N}\right), \ldots H\left(1,\tfrac{j}{N}\right) \ .$$

By our choice of Δ these broken lines all lie in U, and the constituent segments all lie in suitably small open disks inside U.

We claim that it will suffice to prove that the line integrals over the curves $X(0,j-1)$ and $X(m,j)$ are equal for all j and for each j the corresponding line integrals over the curves $X(i-1,j)$ and $X(i,j)$ are equal. As before it will follow that the line integrals over all the broken line curves $X(i,j)$ have the same value. But $X(m,N)$ is a constant curve, so this value is zero. On the other hand, by construction the curve $X(m,1)$ is formed by concatenating $\mathbf{Lin}\,(\Gamma,\Delta)$ and a constant curve, so this value is also the value of the line integral over $\mathbf{Lin}\,(\Gamma,\Delta)$. But now the Lemma implies that the values of the corresponding line integrals over Γ and $\mathbf{Lin}\,(\Gamma,\Delta)$ are equal, and therefore the value of the line integral over the original curve Γ must also be equal to zero.

The first set of equalities follow from the same sort argument used previously for $W(0,j-1)$ and $W(m,j)$ with the restriction of Γ replaced by the broken line curve with vertices

$$H\left(0,\tfrac{j}{N}\right), \ldots H\left(1,\tfrac{j}{N}\right) \ .$$

To verify the second set of equalities, note that the difference between the values of the line integrals over $X(i,j)$ and $X(i-1,j)$ is given by

$$\int_{C(i,j)} P\,dx \ + \ Q\,dy \ - \ \int_{C'(i,j)} P\,dx \ + \ Q\,dy$$

where $C(i,j)$ is the broken line curve with vertices

$$H\left(t_{i-1},\tfrac{j-1}{N}\right), \quad H\left(t_i,\tfrac{j-1}{N}\right), \quad H\left(t_i,\tfrac{j}{N}\right)$$

and $C'(i,j)$ is the broken line curve with vertices

$$H\left(t_{i-1},\tfrac{j-1}{N}\right), \quad H\left(t_{i-1},\tfrac{j}{N}\right), \quad H\left(t_i,\tfrac{j}{N}\right).$$

By the previously quoted result from multivariable calculus we have

$$\int_{C(i,j)} P\,dx + Q\,dy \quad = \quad \int_{C'(i,j)} P\,dx + Q\,dy$$

for each i and j, and therefore the difference between the values of the line integrals must be zero. Therefore the difference between the values of the line integrals over $X(i,j)$ and $X(i-1,j)$ must also be zero, as required. This completes the proof.■

An example

Suppose now that $U = \mathbf{R}^2 - \{\mathbf{0}\}$ and ω is the closed 1-form

$$\frac{x\,dy \;-\; y\,dx}{x^2 \;+ y^2} \;.$$

We would like to describe the possible values for the line integral

$$\int_\Gamma \omega$$

as Γ ranges over all closed piecewise smooth curves in U.

We shall first consider curves of this type whose initial and final value is the unit vector $\mathbf{e}_1 = (1, 0)$, Since we have

$$\pi_1(S^1, \mathbf{e}_1) \;\cong\; \pi_1(U, \mathbf{e}_1) \;\cong\; \mathbf{Z}$$

and the results of this section show that the value of the line integral only depends upon the class of Γ in the fundamental group, it follows that there are only countably many possible values for the line integral. Furthermore, by the definition of concatenation for curves we have

$$\int_{\Gamma+\Phi} \omega \;\;=\;\; \int_\Gamma \omega \;+\; \int_\Phi \omega$$

it follows that in fact the line integral construction yields a homomorphism from $\mathbf{Z}$ to $\mathbf{R}$. Therefore it is enough to evaluate the line integral on a curve which generates the fundamental group. Of course, the standard generator is the counterclockwise circle

$$\Psi(t) \;\;=\;\; (\cos 2\pi\, t,\; \sin 2\pi\, t)$$

and by a standard exercise in multivariable calculus the value of $\int_\Psi \omega$ in this case is 2π. Therefore we have the following:

The set of all possible values for the line integral $\int_\Gamma \omega$ must be the set of all integral multiples of 2π.

If the initial and final point of a curve is some point $\mathbf{p}$ which is not necessarily $\mathbf{e}_1$, we may retrieve the same conclusion as follows: Let α be a piecewise smooth curve joining $\mathbf{e}_1$ to $\mathbf{p}$. Then we know that the construction sending a closed curve Γ based at $\mathbf{e}_1$ to the closed curve

$$\Gamma^* \;\;=\;\; -\alpha \;+\; \Gamma \;+\; \alpha$$

passes to an isomorphism of groups from $\pi_1(U, \mathbf{e}_1)$ to $\pi_1(U, \mathbf{p})$. Since the line integrals of Γ^* and Γ are equal (why?), it follows that the images of the associated homomorphisms from $\pi_1(U, \mathbf{e}_1)$ and $\pi_1(U, \mathbf{p})$ are also equal, so that the latter also consists of all integral multiples of 2π.■