I.2 : Extending Green's and Stokes' Theorems

(Conlon, § 8.1)

In this section we shall use the contents of `extforms2007.pdf` as needed (with the exception of Section 3 in the latter). This is probably also a good time to look back at Sections I.2 from the 246A and also Section III.2 up to Lemma 1 of the latter. Illustrations for this section appear in the following separate file:

`http://math.ucr.edu/~res/figures0102.pdf`

Objectives

In advanced calculus textbooks, it is easy to find proofs of basic results in vector analysis like *Green's Theorem*, *Stokes' Theorem*, and the *Divergence Theorem* in special cases. For example, it is very easy to derive Green's Theorem in the case of regions defined by standard systems of inequalities

$$a \;\leq\; x \;\leq b, \qquad\qquad g(x) \;\leq\; y \;\leq\; f(x)$$

where g and f are continuous functions such that $g(x) \;<\; f(x)$, at least if $x \neq a,\; b$ (see the first illustration in `figures0102.pdf`). However, as noted in many (most?) advanced calculus texts, the result is true in far more general cases, including regions whose boundaries are given by several closed curves (the second illustration in `figures0102.pdf`). The goal of this section is to discuss some of the tools needed in order to extend the previously mentioned results in vector analysis from simple cases to more general ones.

Change of variables formulas

Most advanced calculus texts do not discuss the role of change of variables formulas in connection with the main theorems of vector analysis. Conceptually, the idea is clear. Suppose that we are given a closed region Ω in the plane whose boundary is given by several curves Γ_i with suitable senses of directions (there is an outermost curve which has a counterclockwise sense, and possibly inner curves which will each have a clockwise sense). Let T be a homeomorphism which is defined on an open set containing Ω such that the coordinate functions of T have continuous partial derivatives of all orders and the Jacobian is always positive. Then the image $T[\Omega]$ will be another region in the plane whose boundary consists of similar curves, each having the same sense as its inverse image (the positivity of the Jacobian is needed to ensure this condition).[1] As before, there are illustrations in `figures0102.pdf`).

[1] The assertion in parentheses depends upon knowing that the sense of a closed curve does not change if the Jacobian is positive and the sense is reversed if the Jacobian is negative. This can be seen fairly directly if T is an invertible linear transformation and Γ is the counterclockwise unit circle. Good examples to consider are the linear transformations $T(x, y) = (x + y, y)$ and $T(x, y) = (x, -y)$.

It is natural to ask whether one can prove directly that Green's Theorem holds for the transformed region and boundary curves if it is known to hold for the original region and boundary curves. In fact, this can be shown using the standard change of variables formula for double integrals and similar results for line integrals, but this is usually not done in advanced calculus texts, mainly because the computations needed to verify such a formula quickly become very messy. However, if one uses differential forms, one can do everything fairly easily as indicated below. To simplify the discussion we shall assume that the boundary of Ω consists of a single curve Γ.

The line integrals over the paths Γ and $T \circ \Gamma$ are related by a simple change of variables argument. In the language of `extforms2007.pdf`, the formula is

$$\int_{\Gamma} T^{*}\theta \quad = \quad \int_{T\circ\Gamma} \theta$$

where $\theta \; = \; P\,dx + Q\,dy$ and $T(u,v) \; = \; (x,y)$ (see the bottom of page 5 in the cited document). By the results of Section 4 in `extforms2007.pdf` translating Green's Theorem into a statement about differential forms, we can rewrite the left hand side as

$$\int_{\Omega} d\,(T^{*}\theta) \quad = \quad \int_{\Omega} T^{*}(d\,\theta)$$

where the equality of the terms follows from the Theorem near the bottom of page 5 in the cited reference. Yet another application of the Change of Variables formula below the statement of that theorem shows that the right hand side of the preceding result is equal to

$$\int_{T[\Omega]} d\,\theta$$

and if we combine all these equations, we see that Green's Theorem holds for the transformed curve and the transformed region.

An example

At this point we shall start using material on simplicial decompositions and simplicial complexes from Section I.2 of the 246A notes.

A standard approach to proving more general versions of Green's Theorem is to combine the change of variables principle with

(i) a nice decomposition of Ω into regions that can be analyzed using simple cases of Green's Theorem.

(ii) the change of variables principle described above.

For example, suppose that we are given the closed region bounded by a Star of David curve. It follows immediately that this closed region has a *simplicial decomposition* in the sense of Section I.2 in the 246A notes (see `figures0102.pdf`); specifically, we can cut the region up into solid triangular regions as illustrated in the picture such that the union of these solid regions is the original subset and the intersection of two solid trianglular regions in the collection is either a common edge or a common vertex of the boundary triangles.

We can then apply Green's Theorem to each of the solid triangular regions, and thus the integral over the whole region is equal to the line integrals over the boundary curves of the solid triangular regions. However, one quickly sees that the line integrals over the "new" pieces of boundary curves — the pieces that were introduced when one cut up the original region — will cancel each other in pairs, so the sum of the line integrals over the boundary triangles will reduce to the line integral over the original Star of David curve.

Reformulation in terms of simplicial chains

Predictably, we shall be using material from the 246A notes on simplicial chains (Section III.3 of the 246A notes) in the discussion below. In addition, we shall need the following elaboration of the discussion in the first paragraph of Section IV.1 from the same notes:

LEMMA 0. *Let Λ_n be the n-simplex in $\mathbf{R}^n$ whose vertices are $\mathbf{0}, \mathbf{e}_1. \cdots, \mathbf{e}_n$ so that Λ is the set of all $(x_1, \cdots, x_n)$ satisfying $x_j \geq 0$ for all j and $\sum_j x_j \leq 1$. Let $\mathbf{v}_0, \cdots, \mathbf{v}_n \in \mathbf{R}^n$ be points that also are the vertices of an n-simplex $\mathbf{S}$. Then there is an affine map $T : \Lambda_n \to S$ which is $1-1$ onto, and for which the Jacobian of T at each point is positive.*

Proof. Let T_0 be the unique affine homeomorphism which sends $\mathbf{0}$ to $\mathbf{v}_0$ and $\mathbf{e}_i$ to $\mathbf{v}_i$ for all $i \geq 0$. Then T_0 has all the desired properties except perhaps the positivity of the Jacobian. In any case we know that the Jacobian is everywhere positive or everywhere negative. If the Jacobian is everywhere positive then we can simply take $T = T_0$. If not, let σ be the map from Λ_n to itself which switches coordinates, and take $T = T_0 \circ \sigma$; it will follow that T has all the required properties.∎

In the spirit of Section IV.1 from the 246A notes we should also observe that there is a standard affine homomorphism between Λ_n and the standard simplex $\Delta_n \subset \mathbf{R}^{n+1}$ with vertices $\mathbf{v}_i = \mathbf{e}_{i+1}$. Specifically, take the affine map which sends $\mathbf{0}$ to $\mathbf{v}_0$ and $\mathbf{e}_i$ to $\mathbf{v}_i$ for $i > 0$.

We can relate the preceding constructions on the Star of David set to the constructions of the 246A notes as follows. As before, let $\Lambda_2 \subset \mathbf{R}^2$ be the solid triangular region with vertices $\mathbf{0} = (0,0)$, $\mathbf{e}_1 = (1,0)$ and $\mathbf{e}_2 = (0,1)$, so that Λ_2 consists of all (x,y) such that $x, y \geq 0$ and $x + y \leq 1$. For each solid triangular region, or 2-*simplex*, α in the Star of David set, by Lemma 0 there is a 1–1 onto affine map T_α fromn Λ_2 onto α such that the Jacobian of T_α is always positive. As in Section IV.1 of the 246A notes, each T_α is naturally associated to a free generator of the ordered simplicial chain group $C_2(P)$, where P is the closed Star of David region. The double integral over P in Green's Theorem is a sum of the double integrals over the images of the mappings T_α. By the Change of Variables formulas, this may be viewed as a sum of double integrals over the standard 2-simplex Λ_2 corresponding to the changes of variables given by the maps T_α. Stretching the language still further, we can think of the original double integral as a sum of double integrals determined by the ordered simplicial chain $\sum_\alpha T_\alpha \in C_2(P)$.

Suppose we now apply Green's Theorem to each of the summands. For each summand the associated double integral is equal to a line integral over a path which traces the boundary. Now the definition of simplicial chains and boundaries associates to each 2-simplex a 1-chain called the boundary; specifically, if we let ∂_0, ∂_1 and $\partial 2$ be the straight line segment curves which go from $\mathbf{e}_1$

to $\mathbf{e}_2$, from $\mathbf{0}$ to $\mathbf{e}_2$ and $\mathbf{0}$ to $\mathbf{e}_1$ respectively (so that ∂_i gives the side opposite the i^{th} vertex), then the boundary 1-chain

$$d(\Lambda_2) \quad = \quad \partial_0 \; + \; (-\partial_1) \; + \; \partial_2$$

describes a parametrization for the boundary curve of Λ_2 in the counterclockwise sense, and hence we may view each double integral over Λ_2 as an appropriate line integral over the curve described by $d(\Lambda_2)$.

If we apply this to the chain $\sum_\alpha \; T_\alpha$, we see that the original double integral over the Star of David region is equal to the following sum of line integrals:

$$\sum_\alpha \; \int_{\partial_0 T_\alpha} \omega \; - \; \int_{\partial_1 T_\alpha} \omega \; + \; \int_{\partial_2 T_\alpha} \omega$$

Symbolically, we may view this as a line integral over the simplicial chain

$$d\left(\sum_\alpha \; T_\alpha\right) \quad \in \quad C_1(P) \; .$$

An inspection of the picture suggests that this chain can be simplified dramatically. Namely, the algebraic boundary consists only of a collection of straight line segments which define the topological boundary curve of the original region in the counterclockwise sense (the latter relies on our assumptions about positive Jacobians). All the extra straight line curves from the terms dT_α which do not lie entirely in the boundary turn out to cancel each other in pairs. This means that the line integral over the boudary chain described above must be equal to the line integral of the boundary curve for the original region, where as before the sense of the boundary is counterclockwise. This completes the derivation of Green's Theorem from the special case for trianglular regions and the Change of Variables principle, at least for our example of the Star of David region. The same considerations work for an arbitrary region which has a simplicial decomposition and a boundary which is a simple closed curve as above.

In fact, similar considerations work for regions which have simplicial decompositions, but whose boundaries are unions of simple closed curves. For example, consider the example of a solid square with a square hole in the middle (see `figures0102.pdf` once more). Note that the boundary chain in the figure is a sum of two pieces, one of which corresponds to the outer boundary in the counterclockwise sense and the other of which corresponds to the inner boundary in the clockwise sense. As indicated by one of the drawings, if there are two inner curves, then the boundary splits into a sum of three pieces corresponding to the boundary curves; as before, the outer boundary has a counterclockwise sense and the inner boundary curves both have clockwise senses.

Green's Theorem in more general situations

A general region may have curves composed of pieces that are not straight line curves, and it will be necessary to replace the affine mappings and simplicial chains by more complicated objects. Specifically, we shall want maps T_α which are defined on open sets containing the 2-simplex Λ_2, with continuous partial derivatives and positive Jacobians at every point, and such that the restriction to Λ_2 will be 1–1. Furthermore, we want the images of these mappings to be nonoverlapping in the sense that the image of T_α meets the image of T_β in either (i) the image of a common boundary edge-curve, (ii) the image of a common vertex, and when the intersections are a common edge we want the common boundary curves to have compatible parametrizations. It may not be obvious that all this can be achieved, but in fact it is always possible to do so (however, the proof is definitely nontrivial), and illustrations appear in the file `figures0102.pdf`.

The resulting mappings T_α then have natural interpretations as generators of the group of **singular** 2-chains $\mathbf{S}_2(X)$ defined in Section IV.1 of the 246A notes; here X is the closed region that we are cutting into pieces). Recall that if n is a nonnegative integer, then the group of singular n-chains is a free abelian group with generating set given by continuous mappings T from the standard n-simplex Δ_n in $\mathbf{R}^{n+1}$ (its vertices are the standard unit vectors) to X; we may use the canonical affine homeomorphism from

$$\Lambda_n \quad = \quad \left\{ (x_1, \ \cdots \ , x_n) \in \mathbf{R}^n \mid x_i \ \geq \ 0 \ , \quad \sum_i \ x_i \ \ \leq \ \ 1 \right\}$$

to Δ_n sending $(x_1, \ \cdots \ , x_n)$ to

$$\left(1 \ - \ \sum_i \ x_i \right) \mathbf{e}_1 \ + \ \sum_j \ x_j \mathbf{e}_{j+1}$$

to identify singular n-simplices with continuous mappings defined on Λ_n. Thus the sum of the corresponding singular 2-simplices $\sum_\alpha \ T_\alpha$ may then be viewed as a singular 2-chain, and we can take its boundary using the same formula as in the simplicial case, obtaining an element in $\mathbf{S}_1(X)$. As in the preceding cases, it turns out that the boundary chain simplifies a sum of pieces $\sum_j \ \gamma_j$, where the γ_j's are a nonoverlapping collection of parametrizations for the boundary of X in the counterclockwise sense (and they give the entire boundary curve).

Stokes' Theorem and the Divergence Theorem

Similar considerations work in both cases. For Stokes' Theorem, one must take the smooth mappings T_α such that the normal directions given by

$$\frac{\partial T_\alpha}{\partial x} \times \frac{\partial T_\alpha}{\partial y}$$

are compatible in an appropriate sense. For the Divergence Theorem, there is an added complication of understanding what sorts of 3-dimensional building blocks should be used. It turns out that the right sorts of objects are given by images of the 3-simplex $\Lambda_3 \subset \mathbf{R}^3$ whose vertices are the zero vector and the three standard unit vectors; equivalently, Λ_3 is the set of all $(x, y, z) \in \mathbf{R}^3$ such that $x, y, z \geq 0$ and $x + y + z \leq 1$. Once again, the boundary formula from Section III.2 of the 246A notes is the right one for working with surface integrals over the boundaries.

Final remarks

Clearly the preceding discussion has not given many proofs. In fact, one needs methods from algebraic topology and further input from geometrical topology in order to give rigorous justifications of everything that is said here. Our purpose right now is mainly to provide motivation, giving important links between the algebraic theory of simplicial and singular chains in 246A and familiar issues related to evaluating line, surface and volume integrals by taking complicated sets and cutting them up into relatively simple pieces. Later in these notes we may say more about the machinery needed to give a rigorous justification of the general forms of Green's Theorem, Stokes' Theorem and the Divergence Theorem.