I.3 : Generalized Stokes' Formula

(Conlon, § 8.2)

The main purpose of this section is to strengthen the ties between differential forms from 205C and homological chains from 246A that were discussed in the previous section.

The combinatorial form of Stokes' Theorem (Theorem 8.2.9 on pages 251 – 255 of Conlon) is the fundamental link between the calculus of exterior differential forms and the singular homology theory that was defined and studied in 246A. In the preceding section we examined the 2-dimensional case in considerable detail. This section will concentrate on the most basic aspect of Stokes' Theorem in higher dimensions; namely, its validity in a fundamental special case. We shall be using the basic definitions for chains, differentials and homology in Units III and IV of the 246A notes extensively in this section.

Let q be a nonnegative integer. In 246A we defined a *singular q-simplex* in a topological space X to be a continuous mapping $T : \Delta_q \to X$, where Δ_q is the simplex in $\mathbf{R}^{q+1}$ whose vertices are the standard unit vectors; the group of singular q-chains $\mathbf{S}_q(X)$ was then defined to be the free abelian group on the set of singular q-simplices. The first step in this section is to is to define an analog of these groups involving *smooth* mappings if X is an open subset of $\mathbf{R}^n$ for some n.

Definition. Let q be a nonnegative integer, and as in the preceding section let $\Lambda_q \subset \mathbf{R}^q$ be the q-simplex whose vertices are $\mathbf{0}$ and the standard unit vectors. Also, let U be an open subset of $\mathbf{R}^n$ for some $n \geq 0$. A *smooth singular q-simplex* in U is a continuous map $T : \Lambda_q \to U$ which is *smooth* — in other words, there is some open neighborhood W_T of Λ_q in $\mathbf{R}^q$ such that T extends to a map $W_T \to U$ which is smooth in the usual sense (the coordinate functions have continuous partial derivatives of all orders). The group of *smooth singular q-chains* $\mathbf{S}_q^{\mathrm{smooth}}(U)$ is the free abelian group on all smooth singular q-simplices in U.

There is an obvious natural relationship between the smooth and ordinary singular chain groups which is given by the standard affine isomorphism φ from Δ_q to Λ_q defined on vertices by $\varphi(\mathbf{e}_1) = \mathbf{0}$ and $\varphi(\mathbf{e}_i) = \mathbf{e}_{i-1}$ for all $i > 1$. Specifically, each smooth singular q-simplex $T : \Lambda_q \to U$ determines the continuous singular q-simplex $T \circ \varphi : \Delta_q \to U$. The resulting map of singular chain groups will be denoted by

$$\varphi^{\#} : \mathbf{S}_q^{\mathrm{smooth}}(U) \longrightarrow \mathbf{S}_q(U)$$

with subscripts or superscripts added if it is necessary to keep track of q or U.

One important feature of the ordinary singular chain groups is that they can be made into a chain complex, and it should not be surprising to learn that there is a compatible chain complex structure on the groups of smooth singular chains. We recall the definition of the chain complex structure on $\mathbf{S}_*(X)$ for a topological space X, starting with the preliminary constructions. If Δ_q is the standard q-simplex, then for each i such that $0 \leq i \leq q$ there is an i^{th} face map $\partial_i : \Delta_{q-1} \to \Delta_q$ sending the domain to the face of Δ_q opposite the vertex $\mathbf{e}_{i+1}$ with $\partial_i(\mathbf{e}_j) = \mathbf{e}_j$ if $j \leq i$ and

$\partial_i(\mathbf{e}_j) \;=\; \mathbf{e}_{j+1}$ if $j \geq i+1$. Then each face map ∂_i defines function from singular q-simplices to singular $(q-1)$-simplices by the formula $\partial_i(T) \;=\; T \circ \partial_i$, and the formula

$$d_q \;=\; \sum_{i=0}^{q} (-1)^i \, \partial_i$$

defines a homomorphism from $\mathbf{S}_q(X)$ to $\mathbf{S}_{q-1}(X)$ with some important formal properties given by Theorem III.3.2 and the first two results in Section IV.1 of the 246A notes.

For the analogous constructions on smooth singular chain groups, we first need compatible face maps on Λ_q. Te simplest way to do this is to relabel the vertices of the latter as $\mathbf{0} = \mathbf{v}_0$ and $\mathbf{e}_i = \mathbf{v}_{i+1}$ for all i; then we may define ∂_i^Λ in the same way as ∂_i, the only difference being that we replace the vertices $\mathbf{e}_j$ for Δ_q by the vertices $\mathbf{v}_j$ for Λ_q.

We claim that if $T : \Lambda_q \to U$ is a smooth singular simplex then are all of the faces given by the composites $T \circ \partial_i^\Lambda$; this follows because each of maps ∂_i^Λ is an affine mapping and hence is smooth.

It follows immediately that the preceding constructions are compatible with the simplex isomorphisms φ constructed above, so that $\varphi^\# \circ \partial_i = \partial_i^\Lambda \circ \varphi^\#$, and if we define

$$d_q^{\mathrm{smooth}} : \mathbf{S}_q^{\mathrm{smooth}}(U) \;\longrightarrow\; \mathbf{S}_{q-1}^{\mathrm{smooth}}(U)$$

to be the sum of the terms $(-1)^i \partial_i^\Lambda$, then one has the following compatibility between smooth and singular chains.

PROPOSITION 1. *Let U be an open subset of $\mathbf{R}^n$ for some n, and let $\varphi^\# : \mathbf{S}_q^{\mathrm{smooth}}(U) \to \mathbf{S}_q(U)$ and d_*^{smooth} be the map given by the preceding constructions. Then the latter map makes $\mathbf{S}_*^{\mathrm{smooth}}(U)$ into a chain complex such that $\varphi^\#$ is a morphism of chain complexes.*

The assertion in the first sentence can be verified directly from the definitions, and the first assertion in the second sentence follows from the same sort of argument employed to prove Theorem III.3.2 in the 246A notes. Finally, the fact that $\varphi^\#$ is a chain complex morphism is an immediate consequence of the assertion in the first sentence and the definitions of the differentials in the two chain complexes in terms of the maps ∂_i and ∂_i^Λ.■

We shall denote the homology of the complex of smooth singular chains by $H_*^{\mathrm{smooth}}(U)$ and call the associated groups the *smooth singular homology groups* of the open set $U \subset \mathbf{R}^n$. In the next unit we shall prove the following fundamentally important result.

ISOMORPHISM THEOREM. *For all open subsets $U \subset \mathbf{R}^n$, the associated homology morphism $\varphi_*^\#$ from the smooth singular homology groups $H_*^{\mathrm{smooth}}(U)$ to the ordinary singular homology groups $H_*(U)$.*

Functoriality properties

In order to prove the Isomorphism Theorem, we need to establish additional properties of smooth singular chain and homology groups that are similar to basic properties of ordinary singular chain and homology groups. The first of these is a basic naturality property:

PROPOSITION 2. *Let $U \subset \mathbf{R}^n$, (etc.) be as above, let $V \subset \mathbf{R}^m$ be open, and let $f : U \to V$ be a smooth mapping from U to V (the coordinates have continuous partial derivatives of all orders). Then there is a functorial chain map $f_{\#}^{\mathrm{smooth}} : \mathbf{S}_*^{\mathrm{smooth}}(U) \to \mathbf{S}_*^{\mathrm{smooth}}(V)$ such that $f_{\#}^{\mathrm{smooth}}$ maps a smooth singular q-simplex T to $f \circ T$ and we have the naturality property*

$$f_{\#} \circ \varphi^{\#} \quad = \quad \varphi^{\#} \circ f_{\#}^{\mathrm{smooth}}$$

where $f_{\#}$ is the corresponding map of smooth singular chains from $\mathbf{S}_*(U)$ to $\mathbf{S}_*(V)$.■

COROLLARY 3. *In the setting of the preceding result, one has functorial homology homomorphisms on smooth singular homology, and the maps $\varphi_*^{\#}$ define natural transformations from smooth singular homology to ordinary singular homology.*■

Combining this with the Isomorphism Theorem mentioned earlier, we see that the construction $\varphi_*^{\#}$ determines a natural isomorphism from smooth singular homology to ordinary singular homology for open subsets of Euclidean spaces.

Since we are already discussing functoriality, this is a good point to mention some properties of this sort which hold for differential forms but were not formulated in `extforms2007.pdf`:

THEOREM 4. *Let $f : U \to V$ and $g : V \to W$ be smooth mappings of open subsets in Cartesian (Euclidean) spaces $\mathbf{R}^n$ where n need not be the same for any of U, V, W. Then the pullback construction on differential forms satisfies the identity $(g \circ f)^{\#} = f^{\#} \circ g^{\#}$. Furthermore, if f is the identity on U then $f^{\#}$ is the identity on $\wedge^*(U)$.*

The second of these is trivial, and the first is a direct consequence of the definitions and the Chain Rule for derivatives of composite maps.■

Integration over smooth singular chains

If U is an open subset of $\mathbf{R}^n$ and $T\Lambda_q \to U$ is a smooth singular q-simplex, then the basic integration formula in `extforms2007.pdf` provides a way of defining an integral $\int_T \omega$ if $\omega \in \wedge^q(U)$. There is a natural extension of this to singular chains; if $\mathbf{c}$ is the smooth singular chain $\sum_i n_i T_i$ where the n_i are integers, then since the group of smooth singular q-chains is free abelian on the smooth singular q-simplices the following is well defined:

$$\int_{\mathbf{c}} \omega \quad = \quad \sum_i n_i \int_{T_i} \omega$$

This definition has the following invariance property with respect to smooth mappings $f : U \to V$.

PROPOSITION 5. *Let $\mathbf{c} \in \mathbf{S}_q(U)$, where U is above, let $f : U \to V$ be smooth and let $\omega \in \wedge^q(V)$. Then we have*

$$\int_{f_{\#}^{\mathrm{smooth}}(\mathbf{c})} \omega \quad = \quad \int_{\mathbf{c}} f^{\#}\omega \ .$$

This follows immediately from the definition of integrals and the Chain Rule.■

The combinatorial form of the **Generalized Stokes' Formula** is a statement about integration of forms over smooth singular chains.

THEOREM 6. (Stokes' Formula, combinatorial version) *Let* $\mathbf{c}$, U, ω*...(etc.) be as above. Then we have*

$$\int_{d\mathbf{c}} \omega \;=\; \int_{\mathbf{c}} d\,\omega \;.$$

Full proofs of this result appear on pages 251–253 of Conlon and also on pages 272–275 of Rudin, *Principles of Mathematical Analysis* (3$^{\rm rd}$ Ed.). Here is an outline of the basic steps: First of all, by additivity it is enough to prove the result when $\mathbf{c}$ is given by a smooth singular simplex T. Next, by Proposition 5 and the identity $f^{\#} \circ d \;=\; d \circ f^{\#}$ (see `extforms2007.pdf` for this), we know that it suffices to prove the result when T is the universal singular simples $\mathbf{1}_q$ defined by the inclusion of Λ_q into some small open neighborhood W_0 of Λ_q. In this case the integrals reduce to ordinary integrals in $\mathbf{R}^q$. We can reduce the proof even further as follows: Let $\theta_i \in \wedge^{q-1}(W_0)$ be the basic $(q-1)$-form $dx^{i_1} \wedge \;\cdots\; dx^{i_{q-1}}$, where $i_1 < \;\cdots\; < i_{q-1}$ runs over all elements of $\{1, \;\cdots\;, q\}$ **except** i. By additivity it will suffice to prove Theorem 6 for $(q-1)$-forms expressible as $g\,\theta_i$, where g is a smooth function on W_0. Yet another change of variables argument shows that it suffices to prove the result for $(q-1)$-forms expressible as $g\,dx^2 \wedge \;\cdots\; \wedge dx^q$. Now the exterior derivative of the latter form is equal to

$$\frac{\partial g}{\partial x^1} \cdot dx^1 \wedge \;\cdots\; dx^q$$

so the proof reduces to evaluating the integral of the left hand factor in this expression over Λ_q, and this is done by viewing this multiple integral as an interated integral and applying the Fundamental Theorem of Calculus.∎

A "global" version of Stokes' Formula for arbitrary dimensions is given in Theorem 8.2.3 on page 247 of Conlon.