II. De Rham Cohomology

There is an obvious similarity between the condition $d_{q-1} \circ d_q = 0$ for the differentials in a singular chain complex and the condition $d[q] \circ d[q-1] = 0$ which is satisfied by the exterior derivative maps $d[k]$ on differential k-forms. The main difference is that the indices or gradings are reversed. In Section 1 we shall look more generally at graded sequences of algebraic objects $\{ A^k \}_{k \in \mathbf{Z}}$ which have mappings $\delta[k]$ from A^k to A^{k+1} such that the composite of two consecutive mappings in the family is always zero. This type of structure is called a **cochain complex**, and it is dual to a chain complex in the sense of category theory; every cochain complex determines *cohomology groups* which are dual to homology groups. We shall conclude Section 1 by explaining how every chain complex defines a family of cochain complexes. In particular, if we apply this to the chain complexes of smooth and continuous singular chains on a space (an open subset of $\mathbf{R}^n$ in the first case), then we obtain associated (smooth or continuous) *singular cohomology groups* for a space (with the previous restrictions in the smooth case) *with real coefficients* that are denoted by $S^*(X;\mathbf{R})$ and $S^*_{\mathrm{smooth}}(U;\mathbf{R})$ respectively. If U is an open subset of $\mathbf{R}^n$ then the natural chain maps $\varphi^\#$ from Section I.3 will define associated natural maps of chain complexes from continuous to smooth singular cochains that we shall call $\varphi_\#$, and there are also associated maps of the corresponding cohomology groups. In Section 2 we shall prove that the homology maps $\varphi^\#_*$ and cohomology maps $\varphi^*_\#$ are isomorphisms. This illlustrates a phenomenon which already arose in 246A; namely, there are several different ways to define homology (and cohomology) groups, and each is particularly convenient in certain situations. In Section 3 we shall prove that De Rham cohomology has many of the basic formal properties that hold for singular cohomology. Finally, in Section 4 we prove an important result first discovered by G. De Rham in the 1930s: If U is an open subset of $\mathbf{R}^n$, then the generalized Stokes' Formula from Section I.3 deines a map J from the cochain complex $\wedge^*(U)$ of differential forms on U to the smooth singular cochain complex $S^*_{\mathrm{smooth}}(U;\mathbf{R})$, and **De Rham's Theorem** states that the associated map in cohomology J^* is an isomorphism. Some elementary consequences of this result will also be discussed.

II.1 : Smooth singular cochains

(Hatcher, § 2.1)

We begin by dualizing chain complexes and homology.

Definition. Let R be a commutative ring with unit. A *cochain complex* over R is a pair (C^*, δ^*) consisting of a sequence of R-modules C^q (the cochain modules) indexed by the integers, and coboundary homomorphisms $d^q : C^q \to C^{q+1}$ such that for all q we have $\delta^{q+1} \circ \delta^q = 0$. The *cocycles* in C^q are the elements x such that $\delta(x) = 0$ and the *coboundaries* in C^q are all elements x which are expressible as $\delta(y)$ for some y.

The defining conditions for a cochain complex imply that the image of δ^{q-1} is contained in the kernel of δ^q, and we define the q^{th} **cohomology module** $H^q(C)$ to be the quotient Kernel δ^q/Image δ^{q-1}.

Formally speaking, the notions of chain and cochain complex are categorically dual to each other. Given a chain complex (C_*, d_*), one can define the categorically dual cochain complex

(C^*, δ^*) by the equatiions $C^q = C_{-q}$ and $\delta^q = d_{1-q}$. Cochain complex morphisms can be defined by duality, and one has the following dualizations of standard results for chain complexes and their morphisms:

(1) *Algebraic morphisms of cochain complexes* $f : C \to D$ *pass to algebraic morphisms of cohomology groups* $[f] : H^*(C) \to H^*(D)$.

(2) *The algebraic morphisms in the preceding satisfy the conditions* $[g \circ f] = [g] \circ [f]$ *and* $[\mathrm{id}] = \text{identity}$.

(3) *If we are given an exact sequence of cochain complexes* $0 \to A \to B \to C \to 0$ *(so that one has a short exact sequence* $0 \to A^q \to B^q \to C^q \to 0$*, then there is an associated long exact sequence of homology:*

$$\cdots \to H^{q-1}(C) \to H^q(A) \to H^q(B) \to H^q(C) \to H^{q+1}(A) \to \cdots$$

(4) *The long exact sequence in the previous statement is natural with respect to suitably defined morphisms of short exact sequences of cochain complexes.*

In each case, the proof is a straightforward dualization of the corresponding argument for chain complexes.

Cochain complexes associated to a chain complex

The main reason for introducing formal duals of chain complexes is that there are many situations in which it is necessary to work with both chain complexes and cochain complexes at the same time. The discussion of the Generalized Stokes' Formula in the preceding unit is one basic example. Our next step is to give a general method for constructing many different cochain complexes out of a chain complex.

Definition. Let (S_*, d_*) denote a chain complex over a commutative ring with unit A, and let M be an A-module (we assume all modules satisfy the identity $1 \cdot m = m$ for all m). The *complex of cochains on* S *with coefficient in* M is given by $C^q(S; M) = \mathrm{Hom}_A(S_q, M)$ (*i.e.*, the module of A-homomorphisms), and the coboundary map $\delta^q : C^q(S; M) \to C^{q+1}(S; M)$ is equal to the adjoint map $(d_{q+1})^*$ which takes a cochain (or function) $f : S_q \to M$ into $f \circ d_{q+1}$.

The identity $d_{q+2} \circ d_{q+1} = 0$ implies that $\delta^{q+1} \circ \delta^q = 0$, and therefore we do have a cochain complex

$$(\, C^*(S, M),\ \delta \,)$$

whose cohomology is called the *cohomology of* S *with coefficients in* M and written $H^*(S; M)$.

EXAMPLE. If X is a topological space and $S_*(X)$ is the singular chain complex of X, then for each abelian group M we obtain an associated singular cochain complex with coefficients in M, written $S^*(X; M)$. Likewise, if U is open in some $\mathbf{R}^n$ and $S_*^{\mathrm{smooth}}(U)$ is the smooth singular complex of U, then we have an associated smooth singular cochain complex with coefficients in M, written $S^*_{\mathrm{smooth}}(U; M)$. The Generalized Stokes' Formula implies that the integration map J defines a map of cochain complexes from $\wedge^*(U)$ to $S^*_{\mathrm{smooth}}(U; \mathbf{R})$.

One can now combine the previously described results on formal dualizations with the definitions of associated cochain complexes to obtain the following basic results:

PROPOSITION. *Suppose that $f : S \to T$ is a morphism of chain complexes over the ring A as above, and let M be an A-module as above. Then there are associated morphisms of cochain complexes*

$$f^{\#} : C^*(T; M) \longrightarrow C^*(S; M)$$

and morphisms of cohomology groups

$$f^* : H^*(T; M) \longrightarrow H^*(S; M)$$

which are **contravariantly** *functorial with respect to chain complex morphisms. Furthermore, if $g : M \to N$ is a homomorphism of A-modules, then there are associated morphisms of cochain complexes*

$$g_{\#} : C^*(S; M) \longrightarrow C^*(S; N)$$

and morphisms of cohomology groups

$$g_* : H^*(T; M) \longrightarrow H^*(T; N)$$

which are **covariantly** *functorial in with respect to module homomorphisms.*■

In particular, if we are given a continuous map of topological spaces $f : X \to Y$ and its associated map of singular chain complexes $f_{\#}$, then we obtain maps of singular cochain complexes $f^{\#} : S^*(Y; M) \to S^*(X; M)$ and morphisms of cohomology groups $f^* : H^*(Y; M) \to H^*(X; M)$ which are **contravariantly** functorial with respect to continuous mappings. Likewise, if we are given a smooth map of open subsets in Euclidean spaces $f : U \to V$ and its associated map of smooth singular chain complexes $f_{\#}$, then we obtain maps of singular cochain complexes $f^{\#} : S^*_{\text{smooth}}(Y; M) \to S^*_{\text{smooth}}(X; M)$ and morphisms of cohomology groups $f^* : H^*_{\text{smooth}}(Y; M) \to H^*_{\text{smooth}}(X; M)$ which are **contravariantly** functorial with respect to smooth mappings. Finally, for open subsets in Euclidean spaces the canonical natural transformation from $S^{\text{smooth}}_*(U)$ to $S_*(U)$ defines natural transformations of cochain complexes

$$\varphi^{\#\#} : S^*(U; M) \longrightarrow S^*_{\text{smooth}}(U; M)$$

and cohomology groups $H^*(U; M) \to H_{\text{smooth}}(U; M)$ which are natural with respect to smooth maps.

As in the case of chain complexes, one immediate question is whether the smooth and ordinary definitions of singular chains for open subsets in $\mathbf{R}^n$ yield isomorphic groups. One aim of this unit is to develop enough machinery so that we can prove this, at least in some important special cases. The following question is clearly closely related:

PROBLEM. *Suppose that $f : S \to T$ is a map of chain complexes such that $f_* : H_*(S) \to H_*(T)$ is an isomorphism in homology. Under what conditions is $f^* : H^*(T; M) \to H^*(S; M)$ an isomorphism in cohomology?*

In this section we shall give some frequently occurring conditions under which the cohomology mappings are isomorphims.

We have not yet discussed versions of (3) and (4) for cochain complexes of the form $C^*(S;M)$ because there is a slight complication. If we have a short exact sequence of A-modules $0 \to S \to T \to U \to 0$, then it is fairly straightforward to show that the associated sequence of adjoint homomorphisms

$$0 \longrightarrow \mathrm{Hom}_A(U;M) \longrightarrow \mathrm{Hom}_A(T;M) \longrightarrow \mathrm{Hom}_A(S;M)$$

is exact, but the last map in this sequence is not always surjective. Simple examples can be constructed by taking the short exact sequence $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_2 \to 0$ (in which the self map of $\mathbf{Z}$ is multiplication by 2) and setting M equal to either $\mathbf{Z}$ or $\mathbf{Z}_2$. **However,** if the short exact sequence is **split**, so that there is a map from U to T which yields a direct sum decomposition $T \cong S \oplus U$, then the associated sequence of adjoint homomorphisms will be exact, for the map from $\mathrm{Hom}_A(T;M)$ to $\mathrm{Hom}_A(T;M)$ will then be onto (verify this!). This leads directly to the following result.

PROPOSITION. *Suppose that we are given a short exact sequence of chain complexes* $0 \to S \to T \to U \to 0$ *such that for each* q *the short exact sequence* $0 \to S_q \to T_q \to U_q$ *splits (with no assumptions whether or not the maps* $U_* \to T_*$ *define a chain complex morphism). Then one has a short exact sequence of cochain complexes*

$$0 \longrightarrow C^*(U;M) \longrightarrow C^*(T;M) \longrightarrow C^*(S;M) \longrightarrow 0$$

and associated long exact sequences of cohomology. The latter are contravariantly functorial with respect to morphisms of long exact sequences of (suitably restricted) chain complexes, and they are covariantly functorial with respect to homomorphisms of the coefficient modules.■

This result applies directly to singular cochain complexes. If (X,A) is a pair of spaces with $A \subset X$, then the standard free generators of $S_*(A)$ have a natural interpretation as a subset of the standard free generators for $S_*(X)$, and therefore we have isomorphisms of chain groups $S_q(X) \cong S_q(A) \oplus S_*(X,A)$ for all q (however, such maps **rarely** define an isomorphism of chain complexes). A similar situation holds for smooth singular chains. Therefore, in both cases one has long exact cohomology sequences, and in fact there is a long commutative latter relating these two long exact sequences on the category of open subsets in Euclidean spaces.

The Kronecker index pairing

We have defined cochains to be objects that assign values to every chain, and we would like to have a similar principle in cohomology; namely, if C is a chain complex and M is a module, then a class in $H^q(C;M)$ assigns a value in M to every class in $H_q(M)$. This map turns out to be bilinear, and it is usually called the Kronecker index pairing.

Formally, proceed as follows: Given a cocycle u and a cycle z as above, define $\kappa(u,z) = \langle u,z\rangle \in M$ by choosing f representing u and c representing z and setting $\kappa(u,z) = f(c)$. We are immediately faced with proving the following result to show this is a valid definition.

LEMMA. *In the preceding discussion, if we are given other representatives* $f + \delta\, g$ *and* $c + d(b)$ *for* u *and* z*, then we obtain the same element in* M*.*

Sketch of proof. The value of $[f + gd](c + db)$ is given by

$$f(c) \ + \ gd(c) \ + \ fd(b) \ + \ gdd(b) \ .$$

In this expression the second term vanishes because $d(c) = 0$, the third term vanishes because $fd = 0$, and the final term vanishes because $dd = 0$.■

The following result implies that the Kronecker index is often nontrivial:

PROPOSITION. *If A is a principal ideal domain and C is a chain complex of free A-modules, then the adjoint map $\kappa' : H^q(C; M) \to \mathrm{Hom}_A(H_q(C), M)$ defined by*

$$[\kappa'(u)](z) \quad = \quad \kappa(u, z) \quad \in \quad M$$

is onto.

Proof. Let $B_q \subset C_{q-1}$ denote the image of d_q, and let Z_q be the kernel of d_q. Then we have $C_q/Z_q \cong B_q$. Since we are working over a principal ideal domain, a submodule of a free module is free. Therefore we may deine a one-sided inverse to the projection $C_q \to B_q$ by lifting a set of free generators in B_q to classes in C_q, and then taking the unique extension of this map to a homomorphism of A-modules. This immediately yields a direct sum decomposition

$$C_q \quad \cong \quad Z_q \ \oplus \ B_q$$

and the latter in turn implies that *every homomorphism from Z_q to a module M can be extended to C_q.*

How does this apply to prove the proposition? Suppose that we are given a homomorhism $\alpha : H_q(C) \to M$. Since the domain is the quotient module Z_q/B_{q+1}, it follows that we can pull α back to Z_q and obtain a homomorphism α_0 on the cycles. By the previous paragraph we can extend α_0 to a map α_1 on C_q; this map vanishes on B_{q+1} by construction, and this yields the cocycle condition $\delta(\alpha_1) = \alpha_1 \circ d_{q+1} = 0$. Therefore we conclude that $u = \kappa'([\alpha_1])$.■

There are simple examples to show that κ' is not always onto. Consider the chain complex given by $\mathbf{Z} \to \mathbf{Z}$, where the first copy of the integers is in degree 1 and the map is multiplication by $m > 1$. Then the only nontrivial homology group is $H_0 \cong \mathbf{Z}_m$, and direct computation also shows that the only nonzero cohomology group with integer coefficients is $H^1(C; \mathbf{Z}) \cong \mathbf{Z}_m$. In particular, since the group $\mathrm{Hom}(H_1, \mathbf{Z})$ is trivial, it follows that κ' cannot be injective in this case, and in fact it is the trivial homomorphism. However, the situation is better if we further specialize to chain complexes which are vector spaces over fields.

PROPOSITION. *In the setting of the lemma, if A is a field then κ' is an isomorphism.*

Proof. We already know that the map is onto, so it is only necessary to prove it is 1–1. Suppose now that $\kappa'(u) = 0$ and u is represented by the cocycle $f : C_q \to M$. Then we have $f(c) = 0$ for every cycle $c \in C_q$. This means that f factors into a composite

$$C_q \longrightarrow C_q/Z_q \ \cong \ B_q \longrightarrow M$$

and since we are working with vector spaces over a field we know that the map $B_q \to M$ extends to a homomorphism g on C_q. By construction we know that $f = \delta(g)$, and therefore we conclude that $u = 0$.■

There is a corresponding definition for homology with coefficients in an arbitrary A-module; for the time being we may assume A is an arbitrary commutative ring with unit. To simplify the discussion we shall assume that the chain complex (C_*, d_*) has chain groups C_q which are all free A-modules.

In terms of tensor products, the complex with coefficients is given by $C_* \otimes_A M$; computationally, this means that the elements of $C_q \otimes_A M$ have the form $\sum_i \gamma_i \otimes m_i$ where the γ_i lie in some fixed set of free generators for C_q and the m_i belong to M. The tensor product $a \otimes_A b$ has the standard bilinearity properties. There is an evident definition of mappings $d_q \otimes_A M$, and these make the sequence $C_q \otimes_A M$ into a chain complex, whose homology is called $H_q(C; M)$.

We recall that if P is an A-module and M is a commutative ring with unit such that the map $A \to M$ sending one unit to the other is a ring homomorphism, then $P \otimes_A M$ has a standard structure as an M-module.

In this section we are primarily interested in situations where A is the integers and M is a field; in this case the mappings in the chain complex turn out to be morphisms of vector spaces over fields and the homology groups have associated structures of vector spaces over the field M.

PROPOSITION. *In the setting described above, if M has characteristic zero (no sum of* 1 *with itself finitely many times yields zero), then there is a natural isomorphism of vector spaces from $H_q(C) \otimes M$ to $H_q(C; M)$.*

Proof. First of all, if $z \in H_q(C)$ and $m \in M$, then the mapping in question sends $z \otimes m$ to the class of $c \otimes m$, where c is a cycle representing z. One can check directly that this map is a well-defined and yields a morphism of vector spaces over M. The isomorphism statement follows because tensoring with the characteristic zero field M (in fact with an arbitrary group having no nonzero elements of finite order) sends short exact sequences into exact sequences. The latter means that the cycles in $C_* \otimes M$ are given by $Z_* \otimes M$, the boundaries in $C_* \otimes M$ are given by $B_* \otimes M$, and the homology is given by the quotient of these groups, which is just $H_* \otimes M$.■

COROLLARY. *If S and T are chain complexes of free abelian groups such that the chain map $f : S \to T$ defines an isomorphism in homology, then for each field M of characteristic zero the maps $f^* : H^*(T; M) \to H^*(S; M)$ are also isomorphisms.*

Proof. The tensor product construction takes isomorphisms to isomorphisms, so the map defined by f from $H_*(S) \otimes M$ to $H_*(T) \otimes M$ is an isomorphism. By the proposition it follows that $H_*(S; M)$ to $H_*(T; M)$ is also an isomorphism. Therefore the associated map of dual vector spaces is also an isomorphism. Since the Kronecker index pairing defines a natural isomorphism from the dual space of $H_*(C; M)$ to $H^*(C; M)$ for $C = S \otimes M$ or $T \otimes M$, it follows that these cohomology groups must also be isomorphic under the map defined by f.■

COROLLARY. *If U is an open subset in some Euclidean space, then the natural map from $H^*(U; \mathbf{R})$ to $H^*_{\text{smooth}}(U; \mathbf{R})$ is an isomorphism.*

This is a special case of the preceding result.■