II.2 : Homological comparison theorem

(Hatcher, § 2.3)

The aim of this section is to show that the natural map from smooth singular chains to ordinary chains

$$S_*^{\mathrm{smooth}}(U) \longrightarrow S_*(U)$$

defines isomorphisms in homology and in cohomology with real coefficients if U is an arbitrary open subset of some $\mathbf{R}^n$.

It will be convenient to extend the definition of smooth singular chain complexes to arbitrary subsets of $\mathbf{R}^n$ for some n. Specifically, if $A \subset \mathbf{R}^n$ then the smooth singular chain complex $S_*^{\mathrm{smooth}}(A)$ is defined so that each group $S_q(A)$ is free abelian on the set of continuous mappings $T : \Lambda_q \to A$ which extend to smooth mappings T' from some open neighborhood $W(T')$ of Λ_q to $\mathbf{R}^n$. If A is an open subset of $\mathbf{R}^n$, then this is equivalent to the original definition, for if we are given T' as above we can always find an open neighborhood V of Λ_q such that T' maps V into A.

Clearly the definitions of smooth and ordinary singular chains are similar, and in fact many properties of ordinary singular chain complexes extend directly to smooth singular chain complexes. The following two are particularly important:

(0) If A is a convex subset of $\mathbf{R}^n$ (which is not necessarily open), then the constant map defines an isomorphism from $H_q^{\mathrm{smooth}}(A)$ to $H_q^{\mathrm{smooth}}(\mathbf{R}^0)$ for all q; in particular, these groups vanish unless $q = 0$.

(1) If we are given two smooth maps $f, g : U \to V$ such that f and g are smoothly homotopic, then the chain maps from $S_*^{\mathrm{smooth}}(U)$ to $S_*^{\mathrm{smooth}}(V)$ determined by f and g are chain homotopic.

(2) The construction of barycentric subdivision chain maps $\beta : S_*(U) \to S_*(U)$ in Section IV.4 of the 246A notes, and the related chain homotopy from β to the identity, determine compatible mappings of the same type on smooth singular chain complexes.

The first two of these follow because the chain homotopy constructions from Unit III of the 246A notes send smooth chains to smooth chains. The proof of the final assertion has two parts. First, the barycentric subdivision chain map in Section IV.4 of the 246A notes takes singular chains in the images of the canonical mappings

$$S_*^{\mathrm{smooth}}(W) \longrightarrow S_*(W)$$

into chains which also lie in the images of such mappings. However, the construction of the chain homotopy must be refined somewhat in order to ensure that it sends smooth chains to smooth chains. In order to construct such a refinement, one needs to know that the homology of $S_*^{\mathrm{smooth}}(\Lambda_q)$ is isomorphic to the homology of a point (hence is zero in positive dimensions). The latter is true by Property (0).■

As in the ordinary case, if $\mathcal{W}$ is an open covering of an open set $U \subset \mathbf{R}^n$, then one can define the complex $\mathcal{W}$-small singular chains

$$S_*^{\mathrm{smooth},\mathcal{W}}(U)$$

generated by all smooth singular simplices whose images lie inside a single element of $\mathcal{W}$, and the argument for ordinary singular chains implies that the inclusion map

$$S_*^{\mathrm{smooth},\mathcal{W}}(U) \longrightarrow S_*^{\mathcal{W}}(U)$$

defines isomorphisms in homology. The latter in turn implies that one has long exact Mayer-Vietoris sequences relating the smooth singular homology groups of U, V, $U \cap V$ and $U \cup V$, where U and V are open subsets of (the same) $\mathbf{R}^n$, and in fact one has a long commutative ladder diagram relating the Mayer-Vietoris sequences for (U, V) with smooth singular chains and ordinary singular chains.

The smooth and ordinary singular chain groups for $\mathbf{R}^0$ are identical, and therefore their smooth and ordinary singular homology groups are isomorphic under the canonical map from smooth to ordinary singular homology. By the discussion above, it follows that the canonical map

$$\varphi_*^U : S_*^{\mathrm{smooth}}(U) \longrightarrow S_*(U)$$

is an isomorphism if U is a convex open subset of some $\mathbf{R}^n$. The next step is to extend the class of open sets for which φ_*^U is an isomorphism.

THEOREM. *The map φ_*^U is an isomorphism if U is a finite union of convex open subsets in $\mathbf{R}^n$.*

Proof. Let (C_k) be the the statement that φ_*^U is an isomorphism if U is a union of at most k convex open subsets. Then we know that (C_1) is true. Assume that (C_k) is true; we need to show that the latter implies (C_{k+1}).

The preceding statements about ladder diagrams and the Five Lemma imply the following useful principle: *If we know that φ_*^U, φ_*^V, and $\varphi_*^{U\cap V}$ are isomorphisms in all dimensions, then the same is true for $\varphi_*^{U\cup V}$.* — Suppose now that we have a finite sequence of convex open subsets $W_1, \cdots, W_{k+1}$, and take U and V to be $W_1 \cup \cdots \cup W_k$ and W_{k+1} respectively. Then we know that φ_*^U and φ_*^V are isomorphisms by the inductive hypotheses. Also, since

$$U \cap V \quad = \quad (W_1 \cap W_{k+1}) \cup \cdots \cup (W_k \cap W_{k+1})$$

and all intersections $W_i \cap W_j$ are convex, it follows from the induction hypothesis that $\varphi_*^{U\cap V}$ is an isomorphism in all dimensions. Therefore by the observation at the beginning of this paragraph we know that $\varphi_*^{U\cup V}$ is an isomorphism, which is what we needed in order to complete the inductive step.■

To complete the proof that φ_*^U is an isomorphism for all U, we need the so-called *compact carrier properties* of singular homology. There are two versions of this result.

THEOREM. *Let X be a topological space, and let $u \in H_q(X)$. Then there is a compact subset $K \subset X$ such that u lies in the image of the canonical map from $H_q(K)$ to $H_q(X)$. Furthermore, if K is a compact subset of X, and v and w are classes in $H_q(K)$ whose images in $H_q(X)$ are equal, then there is a compact subset L such that $K \subset L \subset X$ such that the images of v and w are equal in $H_q(L)$.*

Proof. Choose a singular chain $\sum_i n_i T_i$ representing u, where each T_i is a continuous mapping $\Delta_q \to X$. If K is the union of the images $T_i[\Delta_q]$, then K is compact, and it follows that u lies in the image of $H_q(K)$ (because the chain lies in the subcomplex $S_*(K) \subset S_*(X)$.

To prove the second assertion in the proposition, note that by additivity it suffices to prove this when $w = 0$. Once again choose a representative singular chain $\sum_i n_i T_i$ for v; since the image of v in $H_q(X)$ is a boundary, there is a $(q+1)$-chain $\sum_j m_j U_j$ on X whose boundary is $\sum_i n_i T_i$. Let L be the union of K and the compact sets $U_j[\Delta_{q+1}]$; then L is compact and it follows immediately that v maps to zero in $H_q(L)$.■

We shall need a variant of the preceding result.

THEOREM. *Let U be an open subset of some $\mathbf{R}^n$, and let $u \in H_q^{\mathrm{CAT}}(U)$, where CAT denotes either ordinary singular homology or smooth singular homology. Then there is a finite union of convex open subsets $V \subset U$ such that u lies in the image of the canonical map from $H_q^{\mathrm{CAT}}(V)$ to $H_q^{\mathrm{CAT}}(U)$. Furthermore, if V is a finite union of convex open subsets of U, and v and w are classes in $H_q^{\mathrm{CAT}}(V)$ whose images in $H_q^{\mathrm{CAT}}(U)$ are equal, then there is a finite union of convex open subsets W such that $V \subset W \subset U$ such that the images of v and w are equal in $H_q^{\mathrm{CAT}}(W)$.*

Proof. The argument is similar, so we shall merely indicate the necessary changes. We adopt all the notation from the preceding discussion.

For the first assertion, by compactness we know that there is a finite union of convex open subsets V such that $K \subset V \subset U$, and it follows that u lies in the image of the homology of V. For the second assertion, take W to be the union of V and finitely many convex open subsets whose union contains L. It then follows that v maps to zero in the homology of W.■

We can now prove the following general result.

THEOREM. *The map φ_*^U is an isomorphism for arbitrary open subsets of some $\mathbf{R}^n$.*

Proof. If $u \in H_q(U)$, then we know there is some finite union of convex open subsets V such that $u = i_*(u_1)$, where $i : V \subset U$ is inclusion. By our previous results we know that $u_1 = \varphi_*^V(u_2)$ for some $u_2 \in H_q^{\mathrm{smooth}}(V)$, and since $i_* \circ \varphi_*^V = \varphi_*^U \circ i_*$, it follows that $u = \varphi_*^U i_*(u_2)$, so that φ_*^U is onto.

To show that φ_*^U is 1–1, suppose that v lies in its kernel. By the previous results we know that v lies in the image of $H_q^{\mathrm{smooth}}(V)$; suppose that v_1 maps to v. Then it follows that $v_2 = \varphi_*^V(v_1) \in H_q(V)$ maps to zero in $H_q(U)$, so that there is a finite union of convex open subsets W such that $V \subset W$ and v_2 maps to zero in $H_q(W)$. If $j : V \to W$ is inclusion, then it follows that $j_*(v_1)$ lies in the kernel of φ_*^W; however, we know that the latter map is 1–1 and therefore it follows that $j_*(v_1) = 0$. Since the image of the latter element in $H_*^{\mathrm{smooth}}(U)$ is equal to v, it follows that $v = 0$ and hence φ_*^U is 1–1, which is what we wanted to prove.■

If we combine this result with the observations in Section II.1, we immediately obtain a similar result for cohomology with real coefficients:

THEOREM. *If U is an arbitrary open subset of $\mathbf{R}^n$, then the map $\varphi_U^* : H^*(U;\mathbf{R}) \longrightarrow H^*_{\mathrm{smooth}}(U;\mathbf{R})$ is an isomorphism of real vector spaces.*■