II.3 : Eilenberg-Steenrod properties

(Hatcher, §§ 2.1, 2.3, 3.1; Conlon, § 2.6, 8.1, 8.3–8.5)

Definition. Let U be an open subset of $\mathbf{R}^n$ for some n. The **de Rham cohomology groups** $H^q_{\mathrm{DR}}(U)$ are the cohomology groups of the cochain complex of differential forms.

In Section 1 we noted that integration of differential forms defines a morphism J of chain complexes from $\wedge^*(U)$ to $S^*(U;\mathbf{R})$, where U is an arbitrary open subset of some Euclidean space. The aim of this section and the next is to show that the associated cohomology map $[J]$ defines an isomorphism from $H^*_{\mathrm{DR}}(U)$ to $H^*_{\mathrm{smooth}}(U;\mathbf{R})$; by the results of the preceding section, it will also follow that the de Rham cohomology groups are isomorphic to the ordinary singular cohomology groups $H^*(U;\mathbf{R})$. In order to prove that $[J]$ is an isomorphism, we need to show that the de Rham cohomology groups $H^*_{\mathrm{DR}}(U)$ satisfy analogs of certain formal properties that hold for (smooth) singular cohomology. One of these is a homotopy invariance principle, and the other is a Mayer-Vietoris sequence. Extremely detailed treatments of these results are given in Conlon, so at several points we shall be rather sketchy.

The following abstract result will be helpful in proving homotopy invariance. There are obvious analogs for other subcategories of topological spaces and continuous mappings, and also for covariant functors.

LEMMA. *Let T be a contravariant functor defined on the category of open subsets of $\mathbf{R}^n$ and smooth mappings. Then the following are equivalent:*

(1) If f and g are smoothly homotopic mappings from U to V, then $T(f) = T(g)$.

(2) If U is an arbitrary open subset of $\mathbf{R}^n$ and $i_t : U \to U \times \mathbf{R}$ is the map sending u to (u,t), then $T(i_0) = T(i_1)$.

Proof. (1) $\Longrightarrow$ (2). The mappings i_0 and i_1 are smoothly homotopic, and the inclusion map defines a homotopy from $U \times (-\varepsilon, 1+\varepsilon)$ to $U \times \mathbf{R}$.

(2) $\Longrightarrow$ (1). Suppose that we are given a smooth homotopy $H : U \times (-\varepsilon, 1+\varepsilon) \to V$. Standard results from 205C imply that we can assume the homotopy is "constant" on some sets of the form $(-\varepsilon, \eta) \times U$ and $(1-\eta, 1+\varepsilon) \times U$ for a suitably small positive number η. One can then use this property to extend H to a smooth map on $U \times \mathbf{R}$ that is "constant" on $(-\infty, \eta) \times U$ and $(1-\eta, \infty) \times U$. By the definition of a homotopy we have $H \circ i_1 = g$ and $H \circ i_0 = f$. If we apply the assumption in (1) we then obtain

$$T(g) \quad = \quad T(i_1)\circ T(H) \quad = \quad T(i_0)\circ T(H) \quad = \quad T(f)$$

which is what we wanted.■

A simple decomposition principle for differential forms on a cylindrical open set of the form $U \times \mathbf{R}$ will be useful. If U is open in $\mathbf{R}^n$ and I denotes the k-element sequence $i_1 < \cdots < i_k$, we shall write

$$\xi_I \quad = \quad dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

and say that such a form is a *standard basic monomial* k-*forms* on U. Note that the wedge of two standard basic monomials $\xi_J \wedge \xi_I$ is either zero or ± 1 times a standard basic monomial, depending upon whether or not the sequences J and I have any common wedge factors.

PROPOSITION. *Every k-form on U is uniquely expressible as a sum*

$$\sum_I f_I(x,t)\, dt \wedge \xi_I \quad + \quad \sum_J g_J(x,t)\, \xi_J$$

where the index I runs over all sequences $0 < i_1 < \cdots < i_{k-1} \leq n$, the index J runs over all sequences $0 < j_1 < \cdots < j_k \leq n$, and f_I, g_J are smooth functions on $U \times \mathbf{R}$.■

We then have the following basic result.

THEOREM. *If U is an open subset of some $\mathbf{R}^n$ and $i_t : U \to U \times \mathbf{R}$ is the map $i_t(x) = (x,t)$, then the associated maps of differential forms $i_0^\#, i_1^\# : \wedge^*(U \times \mathbf{R}) \to \wedge^*(U)$ are chain homotopic.*

In this example the chain homotopy is frequently called a *parametrix*.

COROLLARY. *In the setting above the maps i_0^* and i_1^* from $H^*_{\mathrm{DR}}(U \times \mathbf{R})$ to $H^*_{\mathrm{DR}}(U)$ are equal.*■

Proof of Theorem. The mappings $P^q : \wedge^q(U \times \mathbf{R}) \to \wedge^{q-1}(U)$ are defined as follows. If we write a q-form over $U \times \mathbf{R}$ as a sum of terms $\alpha_I = f_I(x,t)\, dt \wedge \xi_I$ and $\beta_J = g_J(x,t)\, \xi_J$ using the lemma above, then we set $P^q(\beta_J) = 0$ and

$$P^q(\alpha_I) \quad = \quad \left(\int_0^1 f_I(x,u)\, du \right) \cdot \xi_I \ ;$$

we can then extend the definition to an arbitrary form, which is expressible as a sum of such terms, by additivity.

We must now compare the values of $dP + Pd$ and $i_1^\# - i_0^\#$ on the generating forms α_I and β_J described above. It follows immediately that $i_1^\#(\alpha_I) - i_0^\#(\alpha_I) = 0$ and

$$i_1^\#(\beta_J) \ - \ i_0^\#(\beta_J) \quad = \quad [g(x,1) \ - \ g(x,0)]\, \beta_J \ .$$

Next, we have $d \circ P(\beta_J) = d(0) = 0$ and

$$d \circ P(\alpha_I) \quad = \quad d \left(\int_0^1 f_I(x,u) du) \right) \cdot \xi_I \quad =$$

$$\sum_j \left(\int_0^1 \frac{\partial f_I}{\partial x^j}(x,u)\, du \right) \wedge dx^j \wedge \omega_I \ .$$

Similarly, we have

$$P \circ d(\alpha_I) \quad = \quad P \left(\sum_j \frac{\partial f_I}{\partial x^j}\, dx^j \wedge dt \wedge \xi_I \ + \ \frac{\partial f_I}{\partial t}\, dt \wedge dt \wedge \xi_I \right)$$

in which the final summand vanishes because $dt \wedge dt = 0$. If we apply the definition of P to the nontrivial summation on the right hand side of the displayed equation and use the identity $dx^j \wedge dt = -dt \wedge dx^j$, we see that the given expression is equal to $-d \circ P(\alpha_I)$; this shows that the values of both $dP + Pd$ and $i_1^\# - i_0^\#$ on α_I are zero. It remains to compute $P \circ d(\beta_J)$ and verify that it is equal to $i_1^\#(\beta_J) - i_0^\#(\beta_J)$. However, by definition we have

$$P \circ d(g_J\, \xi_J) \quad = \quad P \left(\sum_i \frac{\partial g_J}{\partial x^i}\, dx^i \wedge \xi_J \; + \; \frac{\partial g_J}{\partial t}\, dt \wedge \xi_J \right)$$

and in this case P maps the summation over i into zero because each form $dx^i \wedge \xi_J$ is either zero or ± 1 times a standard basic monomial, depending on whether or not dx^i appears as a factor of ξ_J. Thus the right hand side collapses to the final term and is given by

$$P \left(\frac{\partial g_J}{\partial t}\, dt \wedge \xi_J \right) \quad = \quad \left(\int_0^1 \frac{\partial g_J}{\partial u}(x,u)\, du \right) \xi_J \quad =$$

$$[\, g(x,1) \; - \; g(x,0)\,]\; \xi_J$$

which is equal to the formuula for $i_1^\#(\beta_J) - i_0^\#(\beta_J)$ which we described at the beginning of the argument.■

COROLLARY. *If U is a convex open subset of some $\mathbf{R}^n$, then $H^q_{\mathrm{DR}}(U)$ is isomorphic to $\mathbf{R}$ if $q = 0$ and is trivial otherwise.*

This follows because the constant map from U to $\mathbf{R}^0$ is a smooth homotopy equivalence if U is convex, so that the de Rham cohomology groups of U are isomorphic to the de Rham cohomology groups of $\mathbf{R}^0$, and by construction the latter are isomorphic to the groups described in the statement of the Corollary.■

COROLLARY. (Poincaré Lemma) *Let U be a convex open subset of some $\mathbf{R}^n$ and let $q > 0$. The a differential q-form ω on U is closed ($d\omega = 0$) if and only if it is exact ($\omega = d\theta$ for some θ).*■

Both of the preceding also hold if we merely assume that U is star-shaped with respect to some point $\mathbf{v}$ (*i.e.*, if $\mathbf{x} \in U$, then the closed line segment joining $\mathbf{x}$ and $\mathbf{v}$ is contained in U), for in this case the constant map is again a smooth homotopy equivalence.■

The Mayer-Vietoris sequence

Here is the main result:

THEOREM. *Let U and V be open subsets of $\mathbf{R}^n$. Then there is a long exact Mayer-Vietoris sequence in de Rham cohomology*

$$\cdots \to H^{q-1}_{\mathrm{DR}}(U \cap V) \to H^q_{\mathrm{DR}}(U \cup V) \to H^q_{\mathrm{DR}}(U) \oplus H^q_{\mathrm{DR}}(V) \to H^q_{\mathrm{DR}}(U \cap V) \to H^{q+1}_{\mathrm{DR}}(U \cup V) \to \cdots$$

and a commutative ladder diagram relating the long exact Mayer-Vietoris sequences for $\{U, V\}$ in de Rham cohomology and smooth singular cohomology with real coefficients.

Proof. The existence of the Mayer-Vietoris sequence will follow if we can show that there is a short exact sequence of chain complexes

$$0 \to \wedge^*(U \cup V) \longrightarrow \wedge^*(U) \oplus \wedge^*(V) \longrightarrow \wedge^*(U \cap V) \to 0$$

where the map from $\wedge^*(U \cup V)$ is given on the first factor by the $i_U^\#$ (where i_U denotes inclusion) and on the second factor by $-i_V^\#$, and the map into $\wedge^*(U \cap V)$ is given by the maps $j_U^\#$ and $j_V^\#$ defined by inclusion of $U \cap V$ into U and V.

The exactness of this sequence at all points except $\wedge^*(U \cap V)$ follows immediately. Therefore the only thing to prove is that the map to $\wedge^*(U \cap V)$ is surjective. This is done using smooth partitions of unity; details are given in Conlon (specifically, the last four lines of the proof for Lemma 8.5.1 on page 267).

The existence of the commutative ladder follows because the Generalized Stokes' Formula defines morphisms from the objects in the de Rham short exact sequence into the following analog for smooth singular cochains:

$$0 \to S^*_{\mathrm{smooth},\mathcal{U}}(U \cup V) \longrightarrow S^*_{\mathrm{smooth}}(U) \oplus S^*_{\mathrm{smooth}}(V) \longrightarrow S^*_{\mathrm{smooth}}(U \cap V) \to 0$$

The first term in this sequence denotes the cochains for the complex of $\mathcal{U}$-small chains on $U \cup V$, where $\mathcal{U}$ denotes the open covering $\{U, V\}$).

Since the displayed short exact sequence yields the long exact Mayer-Vietoris sequence for (smooth) singular cohomology, the statement about a commutative ladder in the theorem follows.■

II.4 : De Rham's Theorem

(Conlon, § 8.9)

The results of the preceding section show that the natural map $[J] : H^*_{\mathrm{DR}}(U) \to H^*_{\mathrm{smooth}}(U; \mathbf{R})$ is an isomorphism if U is a convex open subset of some Euclidean space, and if we compose this with the isomorphism between smooth and ordinary singular cohomology we obtain an isomorphism from the de Rham cohomology of U to the ordinary singular cohomology of U with real coefficients. The aim of this section is to show that both $[J]$ and its composite with the inverse map from smooth to ordinary cohomology is an isomorphism for an arbitrary open subset of $\mathbf{R}^n$. As in Section II.2, an important step in this argument is to prove the result for open sets which are expressible as finite unions of convex open subsets of $\mathbf{R}^n$.

PROPOSITION. *If U is an open subset of $\mathbf{R}^n$ which is expressible as a finite union of convex open subsets, then the natural map from $H^*_{\mathrm{DR}}(U)$ to $H^*_{\mathrm{smooth}}(U; \mathbf{R})$ and the associated natural map to $H^*(U; \mathbf{R})$ are isomorphisms.*

Proof. If W is an open subset in $\mathbf{R}^n$ we shall let ψ^W denote the natural map from de Rham to singular cohomology. If we combine the Mayer-Vietoris sequence of the preceding section with the considerations of Section II.2, we obtain the following important principle:

If $W = U \cup V$ and the mappings ψ^U, ψ^V and $\psi(U \cap V)$ are isomorphisms, then $\psi^{U \cup V}$ is also an isomorphism.

Since we know that ψ^V is an isomorphism if V is a convex open subset, we may prove the proposition by induction on the number of convex open subsets in the presentation $W = V_1 \cup \cdots \cup V_k$ using the same sorts of ideas employed in Section II.2 to prove a corresponding result for the map relating smooth and ordinary singular homology.■

The general case

Most open subsets of $\mathbf{R}^n$ are not expressible as finite unions of convex open subsets, so we still need some method for extracting the general case. The starting point is the following observation, which implies that an open set is a *locally finite* union of convex open subsets.

THEOREM. *If U is an open subset of $\mathbf{R}^n$, then U is a union of open subsets W_n indexed by the positive integers such that the following hold:*

(1) *Each W_n is a union of finitely many convex open subsets.*

(2) *If $|m - n| \geq 3$, then $W_n \cap W_m$ is empty.*

Proof. Results from 205C imply that U can be expressed as an increasing union of compact subsets K_n such that K_n is contained in the interior of K_{n+1} and K_1 has a nonempty interior. Define $A_n = K_n - \mathbf{Int}(K_{n-1})$, where K_{-1} is the empty set; it follows that A_n is compact. Let V_n be the open subset $\mathbf{Int}(K_{n+1}) - K_{n-1}$. By construction we know that V_n contains A_n and $V_n \cap V_m$ is empty if $|n - m| \geq 3$. Clearly there is an open covering of A_n by convex open subsets which are contained in V_n, and this open covering has a finite subcovering; the union of this finite family of convex open sets is the open set W_n that we want; by construction we have $A_n \subset W_n$, and since $U = \cup_n A_n$ we also have $U = \cup_n W_n$. Furthermore, since $W_n \subset V_n$, and $V_n \cap V_m$ is empty if $|n - m| \geq 3$, it follows that $W_n \cap W_m$ is also empty if $|n - m| \geq 3$.■

We shall also need the following result:

PROPOSITION. *Suppose that we are given an open subset U in $\mathbf{R}^n$ which is expressible as a countable union of pairwise disjoint subset U_k. If the map from de Rham cohomology to singular cohomology is an isomorphism for each U_k, then it is also an isomorphism for U.*

Proof. By construction the cochain and differential forms mappings determined by the inclusions $i_k : U_k \to U$ define morphisms from $\wedge^*(U)$ to the cartesian product $\Pi_k \wedge^*(U_k)$ and from $S^*_{\text{smooth}}(U)$ to $\Pi_k\ S^*_{\text{smooth}}(U_k)$. We claim that these maps are isomorphisms. In the case of differential forms, this follows because an indexed set of p-forms $\omega_k \in \wedge^p(U_k)$ determine a unique form on U (existence follows because the subsets are pairwise disjoint), and in the case of singular cochains it follows because every singular chain is uniquely expressible as a sum $\sum_k c_k$, where c_k is a singular chain on U_k and all but finitely many c_k's are zero (since the image of a singular simplex $T : \Delta_q \to U$ is pathwise connected and theopen sets U_k are pairwise disjoint, there is a unique m such that the image of T is contained in U_m).

If we are given an abstract family of cochain complexes C_k then it is straightforward to verify that there is a canonical homomorphism

$$H^*\left(\prod_k C_k\right) \longrightarrow \prod_k H^*(C_k)$$

defined by the projection maps

$$\pi_j : \prod_k C_k \longrightarrow C_j$$

and that this mapping is an isomorphism. Furthermore, it is natural with respect to families of cochain complex mappings $f_k : C_k \to E_k$.

The proposition follows by combining the observations in the preceding two paragraphs.∎

We are now ready to prove the main result:

DE RHAM'S THEOREM. *The natural maps from de Rham cohomology to smooth and ordinary singular cohomology are isomorphisms for every open subset U in an arbitrary $\mathbf{R}^n$.*

Proof. Express U as a countable union of open subset W_n as in the discussion above, and for $k = 0, 1, 2$ let $U_k = \cup_m W_{3m+k}$. As noted in the definition of the open sets W_j, the open sets W_{3m+k} are pairwise disjoint. Therefore by the preceding proposition and the first result of this section we know that the natural maps from de Rham cohomology to singular cohomology are isomorphisms for the open sets U_k.

We next show that the natural map(s) must define isomorphisms for $U_1 \cup U_2$. By the highlighted statement in the proof of the first proposition in this section, it will suffice to show that the same holds for $U_1 \cap U_2$. However, the latter is the union of the pairwise disjoint open sets $W_{3m} \cap W_{3m+1}$, and each of the latter is a union of finitely many convex open subsets. Therefore by the preceding proposition and the first result of this section we know that the natural maps from de Rham to singular cohomology are isomorphisms for $U_1 \cap U_2$ and hence also for $U^* = U_1 \cup U_2$.

Clearly we would like to proceed similarly to show that we have isomorphisms from de Rham to singular cohomology for $U = U_0 \cup U^*$, and as before it will suffice to show that we have isomorphisms for $U_0 \cap U^*$. But $U_0 \cap U^* = (U_0 \cap U_1) \cup (U_0 \cap U_2)$, and by the preceding paragraph we know that the maps from de Rham to singular cohomology are isomorphisms for $U_0 \cap U_1$. The same considerations show that the corresponding maps are isomorphisms for $U_0 \cap U_2$, and therefore we have reduced the proof of de Rham's Theorem to checking that there are isomorphisms from de Rham to singular cohomology for the open set $U_0 \cap U_1 \cap U_2$. The latter is a union of open sets expressible as $W_i \cap W_j \cap W_k$ for suitable positive integers i, j, k which are distinct. The only way such an intersection can be nonempty is if the three integers i, j, k are *consecutive* (otherwise the distance between two of them is at least 3). Therefore, if we let

$$S_m \quad = \bigcup_{0 \leq k \leq 2} W_{3m-k} \cap W_{3m+1-k} \cap W_{3m+2-k}$$

it follows that S_m is a finite union of convex open sets, the union of the open sets S_m is equal to $U_0 \cap U_1 \cap U_2$, and if $m \neq p$ then $S_m \cap S_p$ is empty (since the first is contained in W_{3m} and the second is contained in the disjoint subset W_{3p}). By the first result of this section we know that the maps from de Rham to singular cohomology define isomorphisms for each of the open sets S_m, and it follows from the immediately preceding proposition that we have isomorphisms from de

Rham to singular cohomology for $\cup_m S_m = U_o \cap U_1 \cap U_2$. As noted before, this implies that the corresponding maps also define isomorphisms for U.■

Some examples

We shall now use de Rham's Theorem to prove a result which generalizes a theorem on page 551 of Marsden and Tromba's *Vector Calculus*:

THEOREM. *Suppose that $n \geq 3$ and $U \subset \mathbf{R}^3$ is the complement of some finite set X. If $\omega \in \wedge^1(U)$ is a closed 1-form, then $\omega = df$ for some smooth function f defined on U.*

Proof. It suffices to prove that $H^1_{\mathrm{DR}}(U) = 0$, and by de Rham's Theorem the latter is true if and only if $H^1(U; \mathbf{R})$ is trivial. If X consists of a single point, then U is homeomorphic to $S^{n-1} \times \mathbf{R}$ and the result follows because we know that $H_1(S^{n-1})$ and $H^1(S^{n-1}; \mathbf{R})$ are trivial. We shall prove that $H^1(U; \mathbf{R})$ is trivial by induction on the number of elements in X.

Suppose that X has $k \geq 2$ elements and the result is known for finite sets with $(k-1)$ elements. Write $X = Y \cup \{z\}$ where $z \notin Y$, and consider the long exact Mayer-Vietoris sequence for $V = \mathbf{R}^n - Y$ and $W = \mathbf{R}^n - \{z\}$. Since $V \cup W = \mathbf{R}^n$ and $V \cap W = U$ we may write part of this sequence as follows:

$$H^1(V; \mathbf{R}) \oplus H^1(W; \mathbf{R}) \longrightarrow H^1(U; \mathbf{R}) \longrightarrow H^2(\mathbf{R}^n; \mathbf{R})$$

The induction hypothesis implies that the direct sum on the left is trivial, and the term on the right is trivial because $\mathbf{R}^n$ is contractible. This means that the term in the middle, which is the one we wanted to find, must also be trivial; the latter yields the inductive step in the proof.■

Generalization to arbitrary smooth manifolds

In fact, one can state and prove de Rham's Theorem for every (second countable) smooth manifold, and one approach to doing so appears in Conlon. We shall outline a somewhat different approach here and compare our approach with Conlon's.

The most fundamental point is that one can extend the definition of the differential forms cochain complexes to arbitrary smooth manifolds, and the associated functor for smooth mappings of open subsets of Euclidean spaces also extends to a contravariant functor on smooth maps of smooth manifolds. This is worked out explicitly in Conlon.

Next, we need to know that the cochain complexes of differential forms have the basic homotopy invariance properties described in the previous section. This is also shown in Conlon.

Given the preceding extensions, the generalization of de Rham's Theorem to arbitrary smooth manifolds reduces to the following basic fact:

THEOREM. *If M is an arbitrary second countable smooth n-manifold, then there is an open set U in some Euclidean space and a homotopy equivalence $f : M \to U$.*

This result is an immediate consequence of (i) the existence of a smooth embedding of M in some Euclidean space $\mathbf{R}^k$, which is shown in Subsection 3.7.C of Conlon, (ii) the Tubular Neighborhood Theorem for smooth embeddings, which is Theorem 3.7.18 in Conlon, (iii) the existence of a smooth homotopy equivalence from a manifold to its tubular neighborhood, which is discussed in Example 3.8.12 of Conlon.■

The application of all this to de Rham's Theorem is purely formal. By homotopy invariance the map f defines isomorphisms in de Rham and singular cohomology, and by the naturality of the integration map from de Rham to singular cohomology we have $\psi^M \circ f^* = f^* \circ \psi^U$. Since the maps f^* and ψ^U are isomorphisms, it follows that ψ^M must also be an isomorphism.■

COMPARISON WITH CONLON'S APPROACH. The approaches in Conlon and these notes boh require locally finite open coverings by subsets U_α such that each finite intersection $U_{alpha_1} \cap \cdots \cap U_{\alpha_k}$ is smoothly contractible. For open subsets in some Euclidean space, the existence of such coverings is an elementary observation; in the general case, one needs a considerable amount of differential geometry, including the existence of geodesically convex neighborhoods (Conlon, Section 10.5). Our approach eliminates the need to consider such neighborhoods or to work with Riemann metrics at all; instead, it uses results from Sections 3.7 and 3.8 of Conlon.

One important advantage of Conlon's approach is that it yields additional information about the isomorphisms in de Rham's Theorem. Namely, under these isomorphisms the wedge product on $H^*_{\mathrm{DR}}(M)$, given by the wedge product of differential forms, corresponds to the singular cup product on $H^*(M : \mathbf{R})$ as defined in Chapter 3 of Hatcher. Details are given in Section D.3 of Conlon.