Mathematics 246B, Fall 2007, Take-Home Examination

This will be due on
Thursday, December 13, 2006, at 1:00 P.M.
either in my mailbox, at the front desk of Surge 202
or in an electronic format as indicated below

You must show the work behind or reasons for your answers. Each problem is worth 25 *points.*

1. Let X be a space, and suppose that we are given cohomology classes $a \in H^p(X)$, $b \in H^q(X)$, $c \in H^r(X)$, and $d \in H^t(X)$, Then the cup products $a\,b\,c\,d$ and $d\,c\,b\,a$ satisfy a relation $a\,b\,c\,d \;=\; \varepsilon \cdot d\,c\,b\,a$ where $\varepsilon = \pm 1$. Give an explicit formula for ε in terms of the data described above.

2. Let U be an open set in $\mathbb{R}^n$. If $\mathbf{F}$ is a smooth function from U to $\mathbb{R}^n$ (a *vector field*) with coordinate functions F^j, then its **divergence** is the smooth function given by the standard formula from vector analysis:

$$\nabla \cdot \mathbf{F} \;=\; \sum_j \frac{\partial F^j}{\partial x^j}$$

Construct an $(n-1)$-form ω on U such that $d\omega \;=\; (\nabla \cdot \mathbf{F})\, dx^1 \wedge \cdots \wedge dx^n$, and using de Rham's Theorem show that every smooth function g on U is the divergence of a smooth vector field on U.

3. Given a topological space X, let p_1 and p_2 denote the coordinate projections $X \times X \to X$ onto the first and second coordinates respectively. Suppose now that $X = S^n$ and $H : S^{2k} \times S^{2k} \to S^{2k} \times S^{2k}$ is a homeomorphism. If $\omega_{2k} \in H^{2k}(S^{2k}; \mathbf{Z})$ is a generator, show that $h^* p_1^* \omega_{2k}$ is equal to $\pm\, p_1^* \omega_{2k}$ or $\pm\, p_2^* \omega_{2k}$. [*Hint:* Use cup products and the fact that h^* is an isomorphism in cohomology.]

4. Suppose that X is an arcwise connected space which has an open covering consisting of k contractible open sets, and let $a_1, \cdots, a_k$ be positive-dimensional cohomology classes in $H^*(X; \mathbf{Z})$. Prove that the cup product $a_1 \cdots a_k$ is zero. [*Hint:* Let $U_1, \cdots U_k$ be the covering by contractible open sets, and explain why each a_i lies in the image of $H^*(X, U_i)$. Recall that if V and W are open subsets of Y then one can define a refined cup product from $H^*(Y,V) \times H^*(Y,W)$ to $H^*(Y, V \cup W)$.]

Guidelines for electronic files

Any version of TEX that works easily on the Department's Unix machines (no rare or exotic, hard to find macros) is fine, including standard forms of LATEX. If you wish to send PostScript or `pdf` files, these are also options.

Extra credit problems are on the next page.

Extra credit — Homology, cohomology and covering spaces

1. Suppose that N is a compact connected oriented topological n-manifold, and let M be a Hausdorff q-sheeted covering of N with projection $p : M \to N$. The results of Section 3.G in Hatcher imply that the image of $p_* : H_*(M) \to H_*(N)$ contains the subgroup $q \cdot H_*(N)$ (see the second paragraph on page 321). In particular, this means that $H_n(M) \neq 0$ if $H_n(N) \neq 0$, so that M is orientable if N is.

(i) Let P be a connected topological n-manifold, and let F be a finite subset of P. Prove that the inclusion maps $(P, P - F) \to (P, P - \{x\})$ — where x runs over all elements in F — define an isomorphism from $H_n(P, P - F)$ to $\oplus_x\, H_n(P, P - \{x\})$. [*Hint:* Let W be a union of pairwise disjoint open subsets which are homeomorphic to open n-disks centered at the points of F. Why are the maps $H_n(W, W - F) \to H_n(P, P - F)$ and $\oplus_x\, H_n(W, W - \{x\}) \to \oplus_x\, H_n(P, P - \{x\})$ isomorphisms, and how can this be used?]

(ii) Prove that the map p_* from $H_n(M) \cong \mathbf{Z}$ to $H_n(N) \cong \mathbf{Z}$ is multiplication by q. [*Hint:* Let $y \in N$ and let F denote the inverse image of $\{y\}$. Show that the map from $H_n(M)$ to $H_n(M, M - F)$ takes the generator to the sum of the generators for the codomain with respect to the decomposition in (i). How can one use excision to prove that a generator for each summand in $H_n(M, M - F)$ maps to a generator for $H_n(N, N - \{y\})$ under the map from $(M, M - F)$ to $(N, N - \{y\})$ defined by p. What does this imply about the image of $H_n(M)$ in $H_n(N, N - \{y\})$?]

2. This problem uses the transfer maps in singular homology and cohomology described in Section 3.G of Hatcher. If $p : E \to B$ is a finite covering, this map is constructed using the chain complex of $\mathcal{U}$-small singular chains on B, where $\mathcal{U}$ is an open covering of B by evenly covered open subsets, and the chain complex of $\mathcal{W}$-small singular chains on E, where $\mathcal{W}$ is the set of open subsets of E which map homeomorphically to open subsets in $\mathcal{U}$ under p.

In the setting above, let $\tau^* : H^*(E) \to H^*(B)$ be the transfer map in singular cohomology. Suppose that $x \in H^*(E)$ and $y \in H^*(B)$. Prove that

$$\tau^*\big(\, x \cdot p^*(y)\,\big) \quad = \quad \tau^*(x) \cdot y$$

where as usual multiplication is given by the cup product. [*Hint:* This relation holds on the cochain level, and to prove this it is enough to prove that standard cochain representatives for both sides yield the same value for an arbitrary $\mathcal{U}$-small singular simplex in B.]