FIGURES FOR ALGEBRAIC TOPOLOGY II
LECTURE NOTES

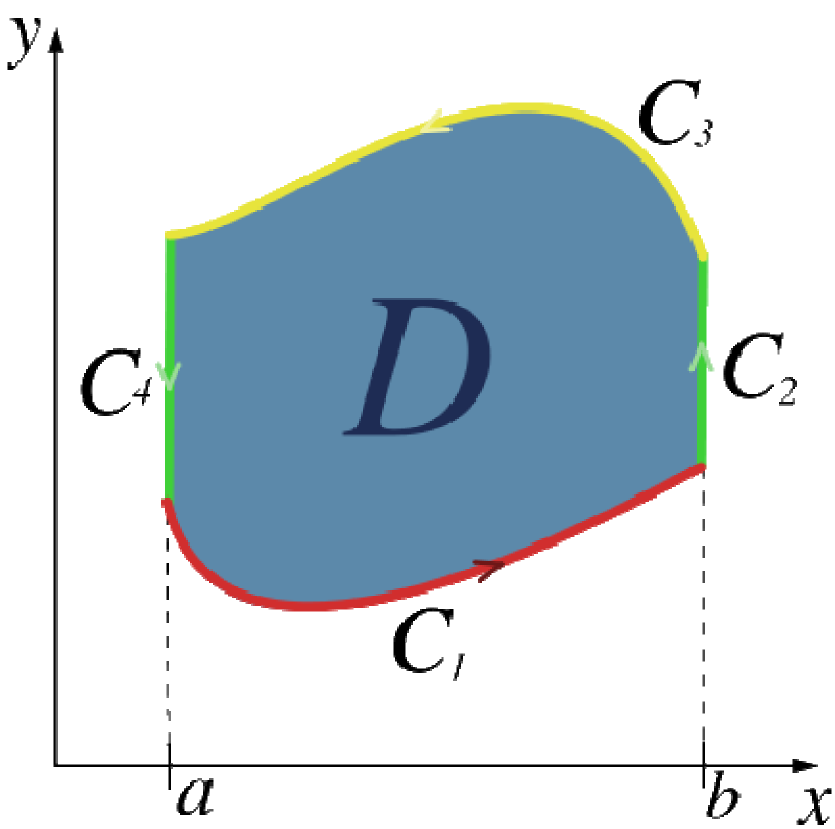
I : Differential forms and their integrals

I.2 : Extending Green's and Stokes' Theorems

The drawings below are related to Section **I.2** of the notes and also to Section **I.2** of the following document from **246A:**

http://math.ucr.edu/~res/algtopfigures.pdf

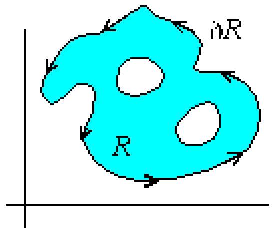
Standard regions. Here is a typical region for which one can find proofs of Green's Theorem in nearly every multivariable calculus textbook. The highlighted set consists of all (x, y) such that $a \leq x \leq b$ and $g(x) \leq y \leq f(x)$, where f and g are continuous functions defined on the closed interval $[a, b]$ and $g(x) < f(x)$ for all x.



(Source: http://en.wikipedia.org/wiki/Image:Green%27s-theorem-simple-region.svg)

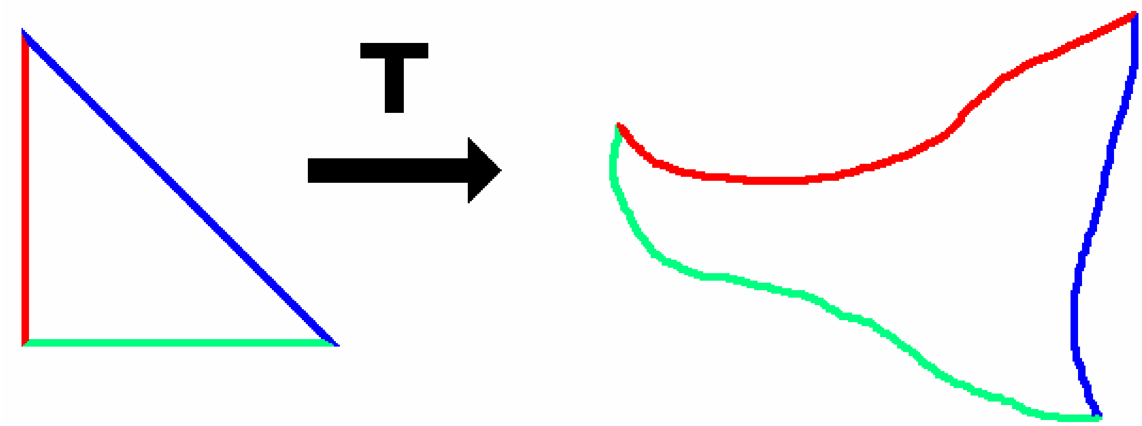
In fact, the proofs in multivariable calculus books also work if the functions f and g are equal at either $x = a$ or $x = b$.

The notes also mention versions of Green's Theorem for which the boundary of the region consists of several curves. The following pictures give an example of such a region.

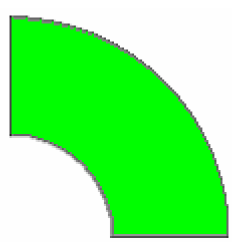


(Source: http://math.etsu.edu/MultiCalc/Chap5/Chap5-4/index.htm)

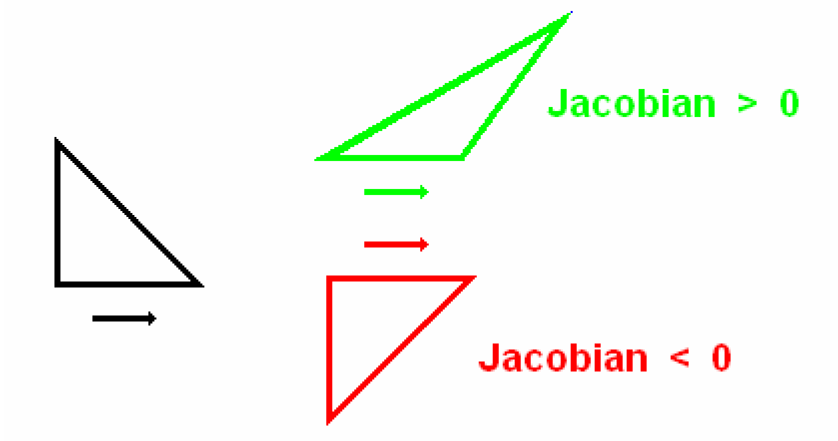
Change of variables. In the notes we mentioned the importance of change of variables formulas for deriving general forms of Green's Theorem. Here is a figure illustrating a typical map **T** on the solid triangular region Λ_2 whose vertices are the zero vector and the two standard unit vectors:



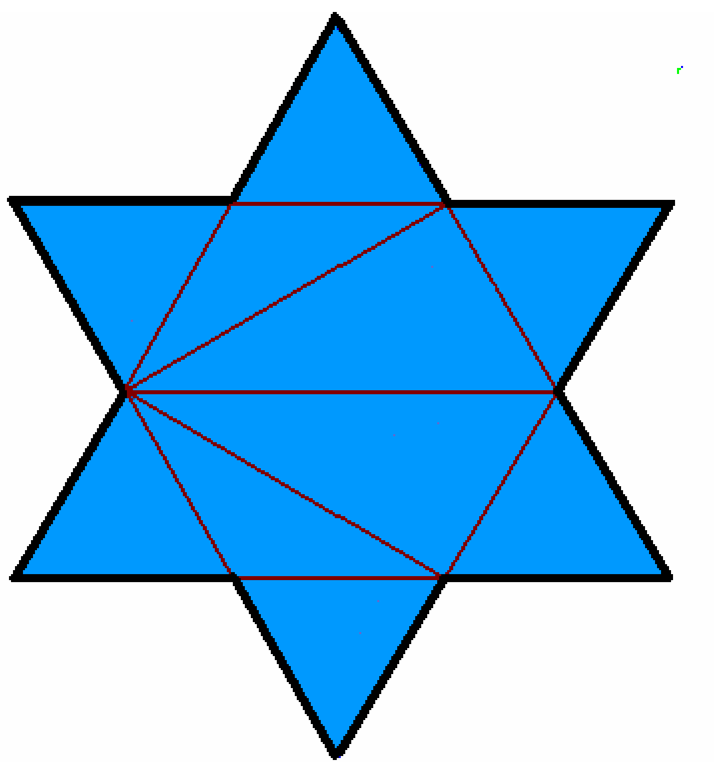
As indicated in the notes, multivariable calculus books generally do not give complete proofs of Green's Theorem for objects like the right hand region in the figure above, but a derivation for such examples is outlined in the notes. On the other hand, multivariable calculus texts frequently do include the special case in which **T** is the polar coordinates transformation and **T** takes a standard region as in the first drawing to a region defined by $a \leq r \leq b$ and $0 < g(r) \leq \theta \leq f(r)$.



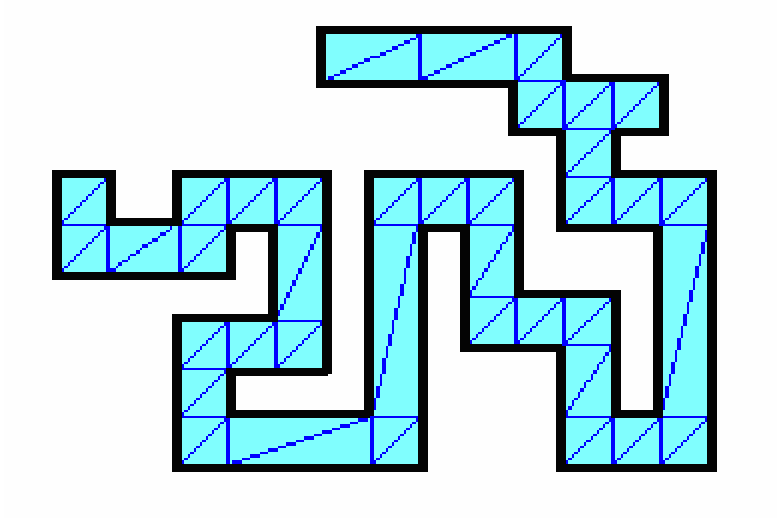
Curve sense and signs of Jacobians. Given a change of variables mapping **T,** if the Jacobian is always positive, then the sense of the image of a simple closed curve $\mathbf{T}[\Gamma]$ is the same as the sense of the original curve Γ, and if the Jacobian is always negative, then the sense of $\mathbf{T}[\Gamma]$ is opposite to the sense of Γ.



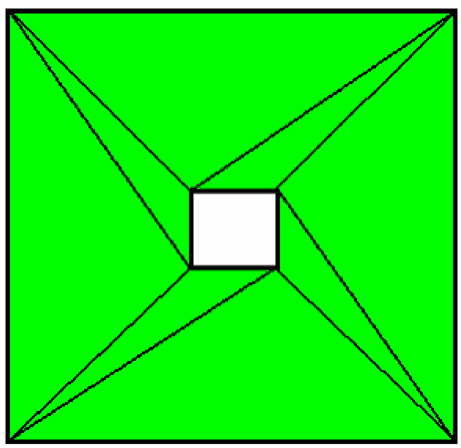
Green's Theorem for nicely triangulated regions. The notes emphasize the case of a region bounded by a Star of David curve as below. If we cut this region up into solid triangular regions as indicated, then the sum of the line integrals over the boundary curves of the triangular regions will reduce to the line integral over the boundary curve of the entire Star of David region; the integrals over the extra curves (in dark red) turn out to cancel each other.

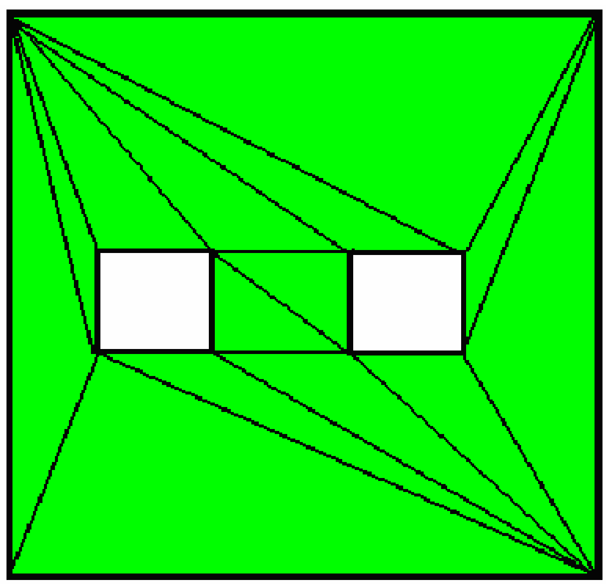


Another example of a region which can be studied in the same way is illustrated below.

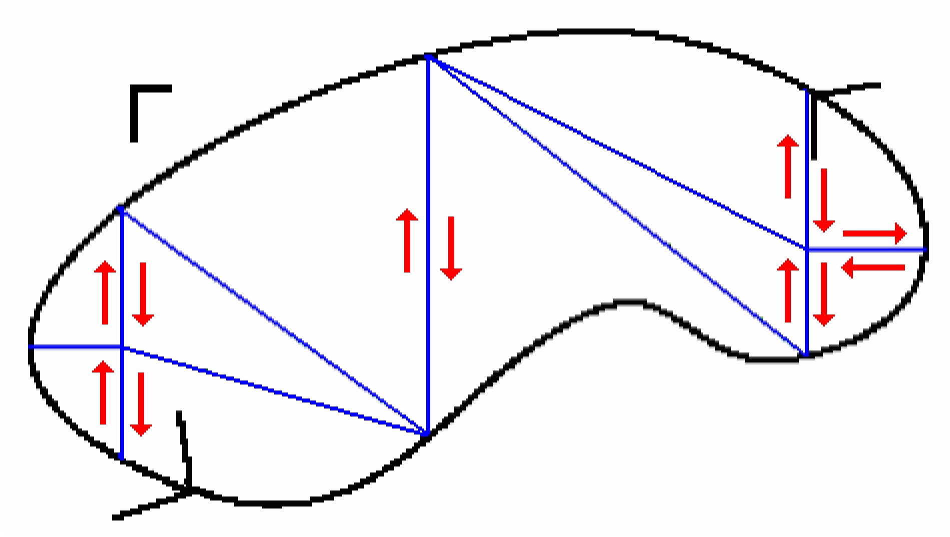


Finally, here are regions that can also be treated by similar methods, but whose boundaries consist of more than one simple closed curve.

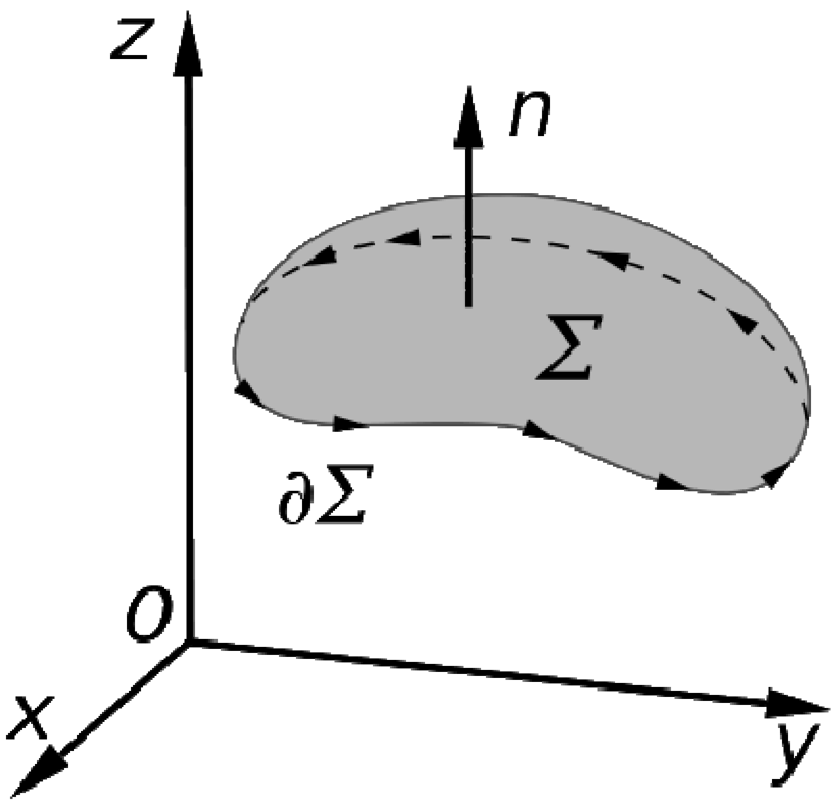




<u>Green's Theorem for fairly general examples.</u> Here is a general sort of region with a single boundary curve and a decomposition into nonoverlapping images of Λ_2 under smooth change of variables maps.

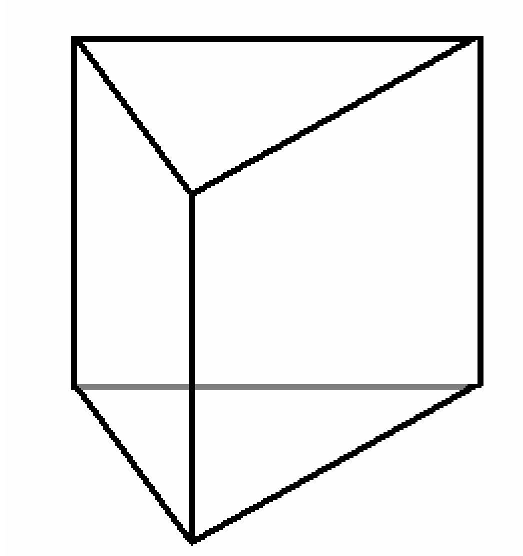


Stokes' Theorem. Here is a typical example involving Stokes' Theorem. It can be studied by splitting the domain into triangular pieces as above.

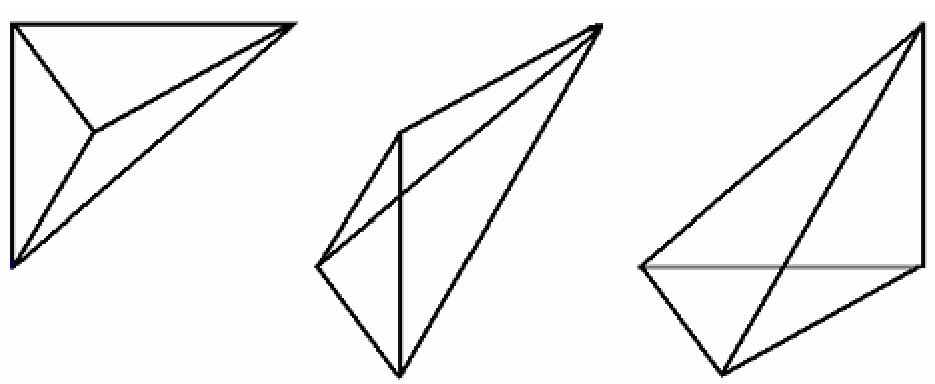


http://en.wikipedia.org/wiki/Image:Stokes%27_Theorem.svg

Divergence Theorem . We shall limit ourselves to discussing the basic sorts of building blocks. Multivariable calculus textbooks prove the result for a class of objects including the standard box shaped regions defined by the three inequalities $a \leq x \leq b$, $c \leq y \leq d$, and $e \leq z \leq f$. One basic question is to determine the proper sorts of "atomic" building blocks. For the plane these were solid triangular regions. In 3 – dimensional space, the proper analogs are solid tetrahedral regions, and in the notation of **246A** these are called 3 – simplices. Basic results from **246A** imply that every solid rectangular region can be expressed as a union of nonoverlapping solid tetrahedral regions. There are two steps to showing this. The first is to note that the solid rectangular region can be split into to solid regions corresponding to triangular prisms; this follows directly from the decomposition of the planar rectangular region in the base into two solid triangular regions. The second step is less obvious, and here are pictures from the 246A file which illustrate the standard decomposition of a solid triangular prism region into solid tetrahedral regions.



A solid triangular prismatic region



$x_0y_0y_1y_2$ $\quad$ $x_0x_1y_1y_2$ $\quad$ $x_0x_1x_2y_2$

Decomposition of the triangular prism into tetrahedra