SIMPLICIAL DUALITY THEORY

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Contents

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PREFACE

This is a revised version of some material I wrote up for limited circulation several years ago. Since that time more modern and efficient approaches to the subject have become readily available, particularly in Spanier's Algebraic Topology. However, there are several places in mathematics (even in current research) where the classical simplicial approach is very helpful, and thus the classical approach is still interesting in its own right aside from its obvious historical and motivational value. On the other hand, it is not entirely trivial to reconstruct a brief, elementary account of classical duality theory out of the literature, and thus I think the following account still has some use.

INTRODUCTION

The original proof of the Poincaré duality theorem depends on two observations, one geometric and one algebraic. The necessary geometric observation is this: Given a simplicial decomposition K of a manifold M via simplices S_i^q ($\dim S_i^q = q$), there is a <u>dual decomposition</u> K^* whose dual cells S_i^q are in 1-1 correspondence with the $(n-q)$-simplices of K, and furthermore this 1-1 correspondence $S \to S^*$ satisfies

$$S^q \subseteq t^p \Leftrightarrow t^{*n-p} \subseteq S^{*n-q} .$$

This tells us that there should be a cell complex A_* for calculating $H_*(M)$ such that $A_q \cong C_{n-q}(K)$ (= alternating simplicial chains). Thus the necessary algebraic observation should be some existence theorem of this sort; furthermore, the existence theorem should be strong enough to tell us that the differentials are assignable so that $H_q(A) \cong H^{n-q}(K) = H^{n-q}(M)$. Since the algebra is of a rather general nature and applies to other situations, it is treated first.

The basic background reference for the notes is Spanier, <u>Algebraic Topology</u>. The reader will also find Eilenberg and Steenrod, <u>Foundations of Algebraic Topology</u>, useful as an alternate reference for the necessary background material. A classical reference for the subject matter is [21]. A deeper discussion of homology manifolds is given in Wilder's book [25], and some important uses of such objects appear in [2], [19] and [26]. Further results on homology manifolds appear in the last chapter of Milnor's notes [17] and in a sequence of papers by Martin and Maunder [11-16]. Duality theorems in more generalized algebraic-topological settings are discussed in Adams' notes [1] and G. Whitehead's paper [24].

1. Abstract subdivision theory.

Given a finite CW complex X, a fundamental theorem of algebraic topology proves the existence of a chain complex C_* of finitely generated free abelian groups such that C_k is free abelian on a set in 1-1 correspondence with the k-cells of X and $H_*(C) \cong H_*(X)$, the singular homology. For our purposes it will be convenient to reformulate the information carried by C_* via more geometrically suggestive objects called abstract cell complexes. The advantage of such objects will be the relative ease of formulating the concept of subdivision for them. A full treatment is given in Lefschetz [10]; we restrict attention to results needed later.

Definition. A (finite) abstract cell complex is a triple $(X, \rho, [\ :\ ])$ consisting of a finite partially ordered set X, a strictly monotonic map $\rho : X \to Z$ and a set function $[\ :\ ] : X \times X \to Z$ such that:

1. $[x:y] = [y:x]$
2. If $[x:y] \neq 0$, then $x > y$ or $x < y$ and $\rho(x) = \rho(y) \pm 1$.
3. If $\rho(x) = \rho(z) + 2$, then

$$\sum_{\rho(y) = \rho(z) + 1} [x:y][y:z] = 0 .$$

$[x:y]$ is called the <u>incidence number</u> of x and y, and $\rho(x)$ is called the <u>dimension</u> of x. The dimension of the cell complex is the maximum value of ρ (such a number exists since X is finite).

We shall employ the common notational convention that x^p designates a cell of X such that $\rho(x^p) = p$.

The most obvious example of a cell complex is a simplicial complex with appropriate incidence numbers of ± 1 (pick a linear ordering of the vertices to eliminate all sign ambiquities). The next class of examples is given by finite CW complexes. Given X, define $\mathrm{Cell}(X)$ to be in 1-1 correspondence with the cells of X, $\rho(e^p) = p$, and define the numbers $[u^p:v^{p-1}]$ by the formula

$$du^p = \sum [u:v]v \qquad \text{in } C_* ,$$

the chain complex for $H_*(X)$.

In keeping with standard practice, the next item concerns morphisms of abstract cell complexes.

<u>Definition</u>. Let X and Y be abstract cell complexes. A <u>morphism</u> $f\colon X \to Y$ is a monotonic function of partially ordered sets such that

1. For some integer k independent of $x \in X$, $\rho(f(x)) = \rho(x) + k$
2. For all $x,y \in X$, $[f(x) : f(y)] = [x:y]$.

k is called the degree of the mapping f. Notice that if $g: Y \to W$ is another morphism of abstract cell complexes, then $g \circ f$ is also a morphism and degree $g \circ f$ = degree g + degree f.

From the above it is clear that an isomorphism of complexes just translates dimensions. Conversely, given any cell complex X and any integer k, we can always find a basically unique isomorphic complex in which dimensions are raised by k units. Any such complex will be called a k-fold suspension of the original complex; two such complexes must be isomorphic by a map of degree zero.

Cell complexes are useful mainly because they give rise to chain complexes in a functorial manner.

Definition. Let X be a cell complex, q an integer. The group of q-chains $C_q(X)$ is the free abelian group on all elements of dimension q. The differential $d_q : C_q(X) \to C_{q-1}(X)$ is given on generators by $d_q x^q = \sum [x^q : y^{q-1}] y^{q-1}$ and extends by linearity. The last condition in the definition of a cell complex implies that $(C(X), d)$ is a chain complex. The homology of X is defined to be the homology of the chain complex. The reader should have no trouble in giving definitions of the cohomology of X and the homology and cohomology with various coefficients.

Of course, if X is a finite CW complex, $C_*(\text{Cell } X) = C_*$, the previously discussed chain complex.

Given a map of cell complexes $f: X \to Y$ of degree k we obtain a map of chain complexes $f_*: C(X) \to C(Y)$ of degree k as follows: on generators $f_*(x^q) = (-1)^{qk}[f(x)]^{q+k}$, and the map extends by linearity. It is clear that this gives the desired functor. As usual, we define the cochain and coefficient functors by $\mathrm{Hom}(C(X),G)$ and $C(x) \otimes G$ respectively, where G is assumed to have grading 0 and Hom takes gradings into their negatives.

Our axioms for a cell complex are dual in the following manner: given a complex X, the dual complex X^* with the reverse ordering, dimension function $\rho^* = -\rho$, and the same incidence numbers as before. X^* is obviously a cell complex, and X^{**} identifiable with X in a natural way. Furthermore $*$ extends to a contravariant functor and $\deg f^* = -\deg f$.

<u>Lemma 1</u>. $H^q(X;G) \cong H_q(X^*;G)$ and $H_q(X;G) \cong H^q(X^*;G)$ all coefficient groups G.

<u>Proof</u>. By duality, only the first needs proof. But $d_q^{X^*}$

$$\begin{array}{ccc} C_q(X^*) & \xrightarrow{d_q^{X^*}} & C_{q-1}(X^*) \\ \| & & \| \\ C_{-q}(X) & \xrightarrow{{}^t d_{-q+1}^X} & C_{-q+1}(X) \end{array}$$

is just the transpose of d_{-q+1}^X in the above identification. (This follows immediately from the definitions).

We next reinforce the analogy between cell complexes and cellular decompositions. Let X be an abstract cell complex, $x \in X$. First a syllabus.

Definitions. The *star* of x, $stx = \{y \in X | x \leq y\}$.

The *closure* of x, $clx = \{y \in X | y \leq x\}$.

The *boundary* of x, $\partial x = clx = \{x\} = \{y | y < x\}$.

The notions of star and closure easily generalize to arbitrary subsets of X by taking unions.

Let $Y \subseteq X$ a subset.

Y is *open* if $x \in X$ implies $stx \subseteq X$.

Y is *closed* if $x \in X$ impliels $clx \subseteq X$.

$Y \subseteq X$ is a *subcomplex* if, with the incidence numbers induced by X, Y is a cell complex in its own right. Notice that by the last property of cell complexes (3) closed and open subsets are subcomplexes. If Y is a closed subcomplex there is an inclusion map $Y \to X$. If U is an open subcomplex there is an inclusion may $X \to U$. Both of these maps have degree zero.

Proposition 1.2. (i) $Y \subseteq X$ is closed iff $U = X - Y$ is open.

(ii) If $U = X - Y$, (Y is closed), the sequence of chain complexes

$$0 \to C(Y) \to C(X) \to C(U) \to 0$$

is exact. Hence there are long exact sequences of homology and cohomology for all coefficient groups .

In keeping with modern notation we shall generally denote U by (X,Y) except when such notation is hopelessly tedious. An example of this would be the open subcomplex stx, which becomes the clumsier $(x, X-\mathrm{st}x)$ in the pair notation.

<u>Definition</u>. Let X be a cell complex, α a set function from X to ± 1. The α-reorientation of X is $(X^\alpha, \rho^\alpha[;])$, where $X^\alpha = X$, $\rho^\alpha = \rho$, $[x:y]^\alpha = \alpha(x)[x:y]\alpha(y)$. If $\alpha(x) = 1(-1)$, α is said to preserve (reverse) the orientation of x. The function α is called a <u>reorientation</u>. If α is a reorientation of X, it determines a chain isomorphism

$$\alpha_* : C(X) \to C(X^\alpha)$$

(<u>Proof</u>: Let $\alpha_* x = \alpha(x)x_\alpha$ where $x_\alpha \in X^\alpha$ is the same as $x \in X$. It is easy to check that α is a chain map. The inverse map is given by $f(x_\alpha) = \alpha(x)x$.) .

In particular, the homology and cohomology of a cell complex are invariant under reorientation.

We next discuss the abstract cell complexes of particular interest to us.

The reader should be familiar with the idea of an augmented chain complex. It is not difficult to see that adding an augmentation to the chain complex of a nonnegative cell complex is the same as adding a single cell in dimension -1 with appropriate incidence numbers. Intuitively, this augmentation cell, as we shall call it, can be thought of as corresponding to the empty set in a cellular decomposition of a space. We shall distinguish an unaugmented complex X from its augmented counterpart by calling the latter X^+. Later on we shall also need R-augmentations, i.e., maps $\varepsilon : C_0(X;R) \to R$ such that $\varepsilon \circ d_1^R = 0$. Of course, Z-augmentations induce R-augmentations for all R.

Definition. Let X be a cell complex, $(n_1, \ldots, n_k)$ any k-tuple of integers. X is $(n_1, \ldots, n_k)$ - cyclic if

$$H_q(X) = \begin{cases} Z & q = n_1, \ldots, n_k \\ 0 & \text{otherwise.} \end{cases}$$

X is acyclic if $H_q(X) = 0$, all q. The reader should have no trouble in defining R-cyclicity and R-acyclicity.

Definition. An augmented complex X^+ is simple if:

1. $\varepsilon(x^0) = \pm 1$, all $x^0 \in X$.

2. For each $x \in X$, $(\mathrm{cl}\,x)^+$ is acyclic. The notation of a R-simple complex is similarly defined.

The best-known example of a simple complex is a simplicial complex. Other examples will appear later in extremely important ways. If X is a topological cell complex, then $(\text{Cell } X)^+$ will be simple if every closed cell is contractible.

Definition. If X is a cell complex, the p-skeleton of $X, X^{(p)} = \{x \in X | \rho(x) \leq p\}$. $X^{(p)}$ is a closed subcomplex of X.

We next derive some elementary properties of simple complexes. In view of previous remarks, we can assume the augmentation maps each cell into 1 by considering a reorientation.

(1.3). If X is acyclic with unique $n+1$ cell x, $X^{(n)}$ is (n)-cyclic and has generating cycle dx in dimension n.

Proof. We want $H_n(X^{(n)}) \cong Z$. But dx and all multiples are nonbounding cycles. Since any n-cycle is an X boundary, all n-cycles over $X^{(n)}$ are of the form qdx for some $q \in Z$.

(1.4). If X is simple, every p-cell in X has a (p-1)-cell in its boundary (and by induction at lease one 0-cell). Every 1-cell has two vertices.

Proof. Suppose the first part is false. Then $dx^p = 0$ implies x^p is a boundary in $(\text{cl}x^p)^+$ — impossible. Thus some $[x^p:x^{p-1}] \neq 0$. If x^1 has only one incident 0-cell, say x^0, then $dx^1 = mx^o$. But this means

$$0 = \varepsilon dx^1 = \varepsilon mx^o = m$$

which again contradicts the acyclicity of $(\text{cl}x^1)^+$. Now assume that

$x^o, y^o, z^o < x^1$. Since $d(x^o - y^o) = d(x^o - z^o) = 0$, there are integers m, n such that $x^o - y^o = mdx^1$ and $x^o - z^o = ndx^1$. Hence $m(x^o - z^o) = n(x^o - y^o)$. Since the chain groups are free abelian, this means $m = n$ and $y^o = z^o$. In other words, there are no more than two distinct vertices.

(1.5). If X is simple and $p \geq 1$, then $(\partial x^p)^+$ is $(p-1)$ cyclic with generating cycle dx^p.

Proof. $(clx^p)^+$ is acyclic and $\partial x^p = [clx^p]^{\pm(p-1)}$. Apply (3.1).

(1.6). In a 1-dimensional simple complex all incidence numbers are $0, \pm 1$.

Proof. $dx^1 = ma + nb$ and $0 = \varepsilon dx^1 = m + n$. Hence $m = -n$. But $\varepsilon(a-b) = 0$ and hence

$$a - b = pdx^1 = pm(a-b).$$

But this implies $pm = 1$, or $m = \pm 1$.

Simple complexes are useful because the acyclicity property guarantees that certain acyclic carrier arguments are valid. We first give the necessary facts on acyclic carriers.

Definition. A carrier function F from the cell complex X to the cell complex Y is a function $X \to P(Y)$ (power set of Y). It is closed if $F(clx) = clF(x)$ all $x \in X$; it is acyclic if $F(clx) \subseteq clF(x)$ and $clF(x)$ is acyclic. The concept of an R-acyclic carrier is defined similarly. The converse carrier function $F^{-1}: Y \to X$ is determined by the rule $x \in F^{-1}(y)$ iff $y \in F(x)$.

Notice that $F^{-1}(y) = \{x\}$ all y iff each $y \in F(x)$ for some x and given $x \neq x^o$, $F(x) \cap F(x^o) = \emptyset$. In this case F is called a block decomposition; $F^{-1}F$ is then the identity. This follows from the string of equivalent statements:

(i) $w \in F^{-1}(x)$

(ii) For some y, $y \in F(x)$ and $w \in F^{-1}(y)$.

(iii) $x \in F^{-1}(y)$ and $w \in F^{-1}(y)$.

But $F^{-1}(y)$ is a one-point set.

Henceforth, the term vertex will be used interchangeably with 0-cell.

Theorem 1.7. Suppose F is an acyclic carrier $X^+ \to Y^+$. If for each x^o there is a vertex in $\mathrm{cl}F(x^o)$, then:

(i) There is an augmentation preserving chain map $f: C(X) \to C(Y)$ carried by F (i.e., $F(x^p) \in C_p(\mathrm{cl}F(x^p))$).

(ii) Any two such maps are chain homotopic by a chain homotopy carried by F. (i.e., $D(x^p) \in C_{p+1}(\mathrm{cl}F(x^p))$).

(iii) If for all cells x^p, $\dim F(x^p) \leq p$, then f is unique.

Proof. Since the proofs of the first two assentions are standard, (see Eilenberg and Steenrod), we only do the third. But suppose there exist $a_1, a_2 \in C_p(\mathrm{cl}F(x^p))$ such that $d_p a_i = f_{p-1} d_p x^p$.

But then $a_1 = a_2$ is a p-cycle and hence $dc_{p+1} = a_1 - a_2$, some $c_{p+1} \in C_{p+1}(\mathrm{cl}F(x^p))$. By assumption the latter group is 0 and thus $dc_{p+1} = 0$; i.e., $a_1 = a_2$.

The analogous statement involving R-notions is also true.

Finally, we can discuss subdivisions.

<u>Definition</u>. Y is a <u>subdivision</u> of X if there is a carrier function $S: X \to Y$ and chain maps σ, τ carried by S, S^{-1} such that:

1. F is a closed block decomposition.

2. $\tau\sigma = 1_{C(x)}, \sigma\tau \simeq 1_{C(Y)}$, the chain homotopy being carred by SS^{-1}.

The notion of R-subdivision is defined analogously.

<u>Proposition 1.8</u>. (i) $\sigma_*: H_*{}^*X;R) \cong H_*(Y;R)$ and $\sigma^*: H^*(Y;R) \cong H^*(X;R)$, all suitable coefficient groups R.

(ii) Suppose (X,X_1) is a closed pair, S a subdivision $X \to Y$. Then $Y_1 = S(X_1)$ is a subdivision of X_1 and (Y,Y_1) is a subdivision of (X,X_1). In particular the long exact sequences of homology and cohomology of the pairs (X,X_1) and (Y,Y_1) are isomorphic.

(iii) A subdivision of a subdivision is a subdivision.

We now joint the ideas discussed in this section, obtaining principal algebraic tool in these notes.

<u>Theorem 1.9</u>. Suppose X^+ and Y^+ are simple complexes and S is a closed acyclic carrier $X \to Y$ satisfying:

(i) S is a block decomposition.

(ii) $S(x^o) = \{y^o\}$ all $x^o \in X$.

Then S determines a subdivision of X.

Proof. First notice SS^{-1} carries the identity. For $w \in SS^{-1}(y)$ iff for some $z \in X$, $w \in S(z)$ and $z \in S^{-1}(y)$. This in turn is equivalent to $w \in S(z)$ and $y \in S(z)$, which is clearly satisfied for $w = y$ since $y \in S(z)$ for some $z \in X$. Also, $S^{-1}S$ is already known to be the identity.

By Theorem 7, there is an augmentation preserving chain map $\sigma: C(X) \to C(Y)$ carried by S. We claim that S^{-1} also satisfies the hypotheses of Theorem 7 and hence gives an augmentation preserving chain map $\tau: C(Y) \to C(X)$.

First of all, $S^{-1}(\mathrm{cl}\,y) \subseteq \mathrm{cl}S^{-1}(y)$. For if $y_o \leq y$, let $S^{-1}(y_o) = \{x_o\}$, $S^{-1}(y) = \{x\}$. Then $y_o \in S(x_o)$ and $y \in S(x)$ imply $y_o \in \mathrm{cl}S(x) = S(\mathrm{cl}\,x)$. Hence $x_o \in \mathrm{cl}\,x$, or $S^{-1}(\mathrm{cl}\,y) \subseteq \mathrm{cl}\,x = \mathrm{cl}S^{-1}(y)$, as desired. That $[\mathrm{cl}S^{-1}(y)]^+$ is acyclic follows from $S^{-1}(y) = \{x\}$. Finally, there is a vertex in $\mathrm{cl}S^{-1}(y)$ by (1.4).

The composites $\sigma\tau$ and $\tau\sigma$ are carried by SS^{-1} and $S^{-1}S$ (= identity). Hence $\sigma\tau$ is homotopic to $1_{C(Y)}$ by a chain homotopy carried by SS^{-1}. On the other hand, $\dim(\mathrm{cl}S^{-1}S(x^p)) = p$ and $1_{C(X)}$ is obviously carried by $S^{-1}S$. Hence by the third part of Theorem 7, $\tau\sigma = 1_{C(X)}$.

For simplicial complexes, the most familiar subdivion is the barycentric subdivision. The same constructions go through for simple complexes.

Definition. Let X be an abstract cell complex; form the simplex $\Delta(X)$ whose vertices are elements of X. The derived complex ${}^1X$ (or X^1) consists of the subcomplex of $\Delta(X)$ containing all simplices $x_o \ldots x_p$ such that $x_o < \ldots < x_p$. Inductively we define ${}^nX = {}^1({}^{n-1}X)$. Unless otherwise stated, when the vertices of a simplex in ${}^1X$ are listed, it is assumed that they are in the canonical order given by X.

It should be clear that, in general, ${}^1X$ is not a subdivision of X. (Exercise: find examples). However, we can still make further constructions in analogy with simplicial complexes. The derived carrier $S: X \to {}^1X$ is given recursively by $S(x^o) = x^o$ and $S(x^p) = S(\partial x^p) * x^p = \{y_o \ldots y_q x^p | y_o < \ldots < y_q < x^p\}$. Thus S maps onto ${}^1X$ and $x \neq x'$ implies $S(x) \cap S(x') = \phi$. S is closed by the following string of equivalences:

(i) $s \in cl(Sx^p)$.

(ii) $s = y_o \ldots y_q$ where $y_q \leq x^p$.

(iii) $y_q \in clx^p$.

(iv) $y_o \ldots y_q \in S(clx^p)$.

Theorem 1.10. If X is simple, ${}^1X$ is a subdivision of X.

Proof. In view of Theorem 9, we only need that $S(cl x^p)^+$ is acyclic. Define a chain homotopy on $C(clS(x^p))$ taking $y_o \ldots y_q$ to $y_o \ldots y_q x^p$, where it is understood that the latter element is zero if $y_q = x^p$. Since this chain homotopy is clearly a contracting one, S is acyclic .

We conclude by stating some special forms of the theorem. Let X be a simplicial complex, σ, τ the chain maps $C(X) \to C(Y)$ and $C(Y) \to C(X)$ respectively.

Corollary 1.11. Let $\pi\colon {}^1X \to X$ be a simplicial map taking $x \in {}^1X$ into some chosen vertex $A < x$. Then $\pi_* = \tau_*$ and $\pi^* = \tau^*$.

Proof. It is easy to check that π_* is a two sided homotopy inverse for σ by acyclic carrier augments. Since two-sided inverses are unique, $\pi_* = \tau_*$ and $\pi^* = \tau^*$.

For the next result, we assume that an ordering has been chosen for the vertices of X. Define a chain map $\eta\colon C(X) \to C({}^1X)$ as follows: If $A_o \ldots A_q$ is a simplex in X and $A_o < \ldots < A_q$, then $\eta(A_o \ldots A_q) = (A_o)(A_oA_1)\ldots(A_oA_1\ldots A_q)$. The following result will be needed later:

Corollary 1.12. $\eta_* = \sigma_*$ and $\eta^* = \sigma^*$.

Proof. Let the map π given in corollary 11 take x into its last vertex. It is clear that $\eta_*\pi_* = 1_{C(X)}$. Hence η_* is a one-sided inverse for π_*. Since σ_* is a two-sided inverse for π_* we must have $\eta_* = \sigma_*$. Similar considerations hold for η^* .

2. The dual decomposition.

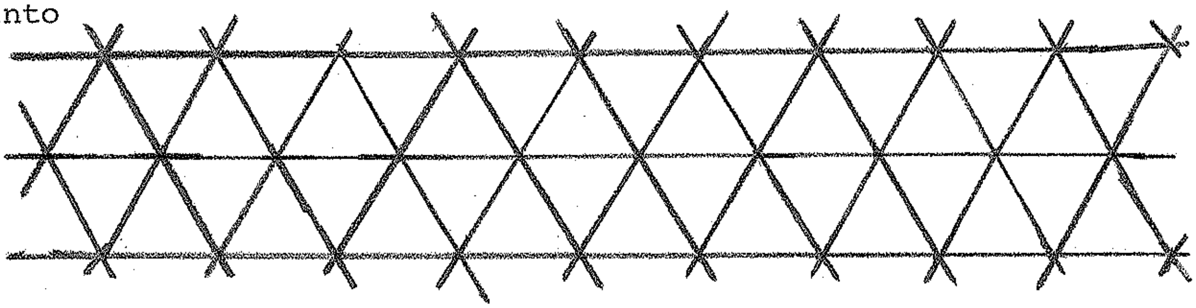
The study of duality in polyhedra was undertaken by nineteenth century mathematicians as a natural extension of the concept of duality in projective 3-space. In particular, given a polyhedron, its dual was to have vertices in 1-1 correspondence with faces of the original one, similarly for faces and vertices reversed, and edges in 1-1 correspondence with those of the original polyhedron. Moreover, the double dual was to be equivalent to the original polyhedron.

Perhaps the easiest examples of this duality are supplied by the five regular polyhedra. Given one of these objects, the center points of the faces form the vertices of a new polyhedron, and through each old vertex there is a line perpendicular to a unique new face that meets the face in its center. Edges are also in 1-1 correspondence such that the line joining the centers of corresponding enges is perpendicular to both of them. Moreover, if this is done twice in succession, a polyhedron similar to the original one is obtained. The duality relationships are summarized below:

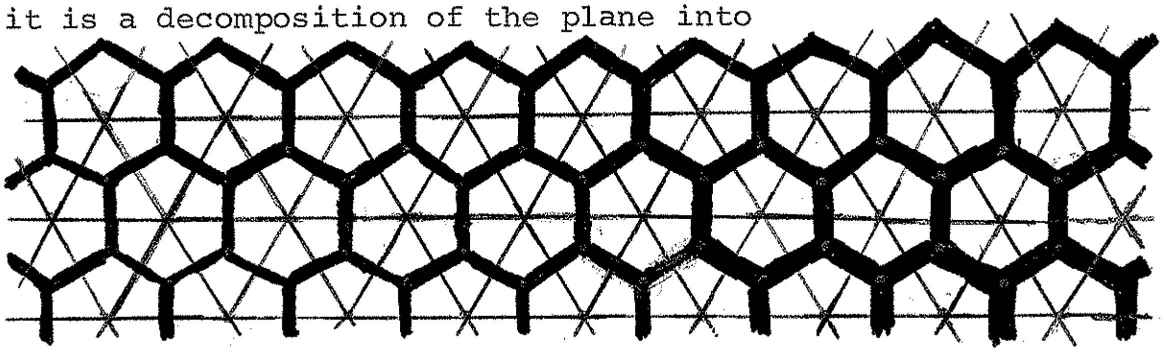
REGULAR POLYHEDRON	DUAL
tetrahedron	tetrahedron
cube	octahedron
dodecahedron	icosahedron

The reader is invited to draw pictures to convince himself of these assertions (See Hilbert and Cohen-Vossen [7],pp.90-92).

For our purposes the most instructive example is the plane, viewed as an infinite polyhedron with a decomposition into



equilateral triangles. The dual of this is easily visualized; it is a decomposition of the plane into



regular hexagons. The correspondence is clear from the above picture (Exercise: write everything out analytically). Each vertex of a hexagon corresponds to the center of a triangle and vice versa. Corresponding sides and specified by one being the perpendicular bisector of the other.

The reader is invited to analyze other regular decompositions of the plane similarly; See Coxeter [5],pp.62,157-158, for more on these and related constructions. A good intuitive discussion of this duality appears in Holden, Spaces, Shapes and Symmetry (Columbia Univ. Press, 1971), especially pp. 5-9,50-51.

Our goal is to give a systematic generalization of such decompositions to arbitrary simplicial complexes.

We shall need a strong version of a result in Spanier (Cor.11,p.124). Namely, suppose that L is a full subcomplex of K and $N(L) = N(L,K)$ is the neighborhood constructed in proving that result.

Then $N({}^1L, {}^1K)$ is a subneighborhood, $\overline{N^1(L)} = \overline{N({}^1L, {}^1K)} \subseteq N(L)$, and the strong deformation retraction sends points of $\overline{N^1(L)} \times I$ into $\overline{N^1(L)}$. In particular, L is a strong deformation retract of $\overline{N^1(L)}$. A complete proof may be found in Eilenberg and Steenrod, pp.70-72.

<u>Definition</u>. Let K be a locally finite complex, $A_o \ldots A_q$ a simplex in K. The cosimplex $A_o^* \ldots A_q^* = \bigcap_{i=0}^{q} \overline{N^1(A_i)}$; it is obviously a subcomplex of ${}^1K$.

<u>Theorem 2.1</u>. If b is the barycenter of $A_o \ldots A_q$, then $A_o^* \ldots A_q^*$ consists of all simplices $bz_i \ldots z_r$ in ${}^1K$ and their faces.

<u>Corollary 2.2</u>. $A_o^* \ldots A_q^*$ is contractible to b and hence acyclic .

<u>Proof</u>. $\overline{N^2(A)}$ consists of all faces of simplices in $N^2(A)$; i.e., all faces of simplices of the form $As_1 \ldots s_r$. It follows that $y_1 \ldots y_r$ is a simplex of $A_o^* \ldots A_q^*$ if and only if all $A_i y_1 \ldots y_r$ are simplices of ${}^2K$. This means each $A_i \in y_1$ in

K, so that $A_o \ldots A_q \subseteq y_1$. Hence $y_1 \ldots y_r$ is a face of $bz_1 \ldots z_s$ for some z's, and it is a proper face if and only if the inclusion of $A_o \ldots A_q$ in y_1 is proper. Conversely, anything of the form $bw_1 \ldots w_t$ is in $A_o^* \ldots A_q^*$ since it is a face of all simplices $A_i bw_1 \ldots w_t$ if $q > 0$; but the case $q = 0$ was considered in the first sentence of the proof .

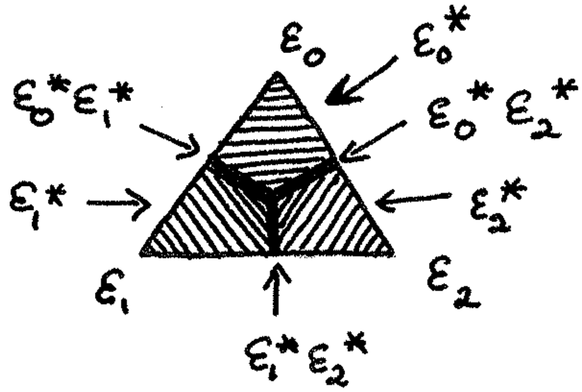
Definition. K is homogeneous of dimension n if K is n-dimensional and every simplex is contained in an n-simplex.

Corollary 2.3. If K is homogeneous of dimension n, then the r-cosimplices of K have dimension n-r.

Proof. Apply the formula of Theorem 1 to see that simplices of dimension n-r and no greater exist in any r-cosimplex .

Perhaps the most basic duality between simplices and co-simplices is the purely set-theoretic one. Suppose that s and t are simplices in K, and s^* and t^* are their cosimplices. Then $s < t$ if and only if $|t^*| \subset |s^*|$ is immediate from the formula of Theorem 1 ($|Y|$ = underlying topological space of Y). Thus, given a simplicial complex K, we have constructed a polygonal decomposition of $|K|$ whose cells correspond to the simplices of K in a 1-1 order-reversing manner.

Example. Consider the triangle Δ_2. The barycenter is $\varepsilon_o^* \varepsilon_1^* \varepsilon_2^*$, the ε_i^* are the shaded portions, and the $\varepsilon_i^* \varepsilon_j^*$ are the



thick lines from the barycenter to the boundary corresponding to the appropriate intersections. If we decompose the whole plane via equilateral triangles, we get the previously discussed hexagonal decomposition.

The rest of this section is devoted to studying certain subcomplexes of the cosimplicial decomposition.

Definition. Let K be a simplicial complex, L a subcomplex of K. The dual complex to L in K, written $K \overset{*}{-} L$ (called the supplement of L by G. Whitehead [24]), consists of all simplices in the set

$$\{s \mid s \in t^* \text{ and } |t^*| \cap |L| = \phi\}.$$

This is again a subcomplex. $K \overset{*}{-} L$ is also a subcomplex in terms of the dual decomposition, since the condition on the subcomplex can clearly be stated in terms of cosimplices.

In the previous example $\Delta_2 \overset{*}{-} \dot{\Delta}_2$ consists of only the barycenter. Other examples show that the dual complex to a subcomplex can be very badly behaved. However, things simplify if L is a full subcomplex of K, which we shall asume for the rest of this section.

Before proceeding, we notice that if L is a full subcomplex of K, then $\alpha \in L$ if and only if $\sum_{A \in L} \alpha_A = 1$.

Proposition 2.4. $|K \overset{*}{-} L|$ contains all vertices of ${}^1K$ except those in ${}^1L$.

Proof. Any vertex of ${}^1L$ is in $|L|$, and $|K \overset{*}{-} L| \cap |L| = \emptyset$ (under the usual identification of $|S|$ with $|{}^1S|$). Next observe that if A is a vertex of K not in L, then $\overline{N^1(A)} \cap L = \emptyset$. For $\overline{N^1(A)} \subseteq N(A)$ and $N(A) \cap L = \emptyset$ since $a \in N(A)$ iff $\alpha(A) > 0$, in which case $\sum \{\alpha(B) | B \in L\} < 1$. But this is equivalent to $\alpha \notin L$. If b is a vertex of ${}^1K$ not in ${}^1L$, b is a barycenter of $A_0 \ldots A_q$, where some $A_i \notin L$. But $b \in \overline{N^1(A_i)}$, and hence it is a vertex of $K \overset{*}{-} L$.

Proposition 2.5. $K \overset{*}{-} L$ is a full subcomplex of ${}^1K$.

Proof. Suppose $s_0 \ldots s_q$ is a simplex of ${}^1K$, and all s_i are in $K \overset{*}{-} L$ — that is, not in ${}^1L$. Then s_0 is a simplex of K not in L, and hence there is a vertex A of s_0 not in L. It follows that

$$|As_0 \ldots s_q| \subseteq \overline{N^2(A)} \subseteq |K \overset{*}{-} L| \ .$$

Proposition 2.6. $|K \overset{*}{-} L|$ is a strong deformation retract of $|K| - |L|$.

Proof. It will suffice to show that $N(K \overset{*}{-} L) = |K| - |L|$. But $N(K \overset{*}{-} L)$ is the union of the regular neighborhoods of all vertices of ${}^1K$ not in ${}^1L$. The result follows from the equivalences

(i) $a \in N(K \overset{*}{-} L)$

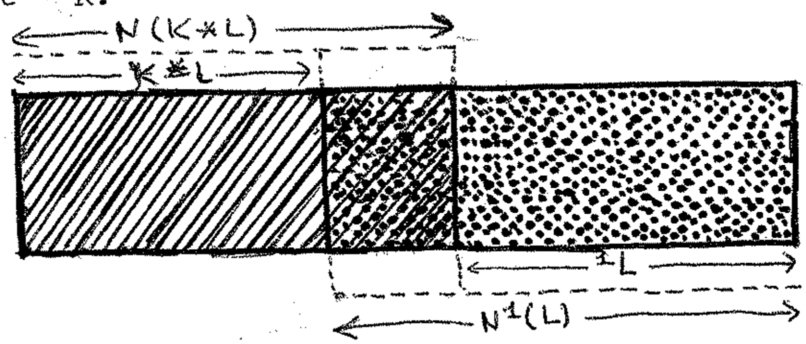
(ii) For some simplex x not on L, the coordinate α_s is positive

(iii) $\sum_{x \in L} \alpha_x < 1$

(iv) $\alpha \in |{}^1L| = |L|$.

It is also true that $N^1(L) = |K| = |K \overset{*}{-} L|$, although we shall not need this fact; this is easily proved using the fact that $K \overset{*}{-} L$ is a full subcomplex.

Here is a picture to show what is going on. The shadings denote the regular neighborhoods, and the box is understood to represent ${}^1K$.



Thus $K \overset{*}{-} L$ is a relative complement of sorts to L (explaining Whitehead's use of the term <u>supplement</u>).

<u>Notation</u>. If Y is a space, ${}_sH_*(Y)$ denotes its singular homology. If (K,L) is a simplicial complex pair, then $\nu_{(K,L)}$ is the natural isomorphism $H_*(K,L)$ (simplicial)$\to {}_sH_*(|K|,|L|)$ defined by viewing the generators of $C_q(K)$ as defining affine linear (hence continuous) maps $\Delta_q \to K$ (see Eilenberg and Steenrod, pp.200-203, or Spanier, Theorem 8, p.171).

Given this notation, the following is an obvious consequence of Proposition 2.6.

Corollary 2.7. There is the following commutative diagram

$$\begin{array}{ccccccccc}
\cdots \to H_{q+1}(K, K \overset{*}{-} L) & \longrightarrow & H_q(K \overset{*}{-} L) & \longrightarrow & H_q({}^1K) & \longrightarrow & H_q({}^1K, K \overset{*}{-} L) & \longrightarrow & \cdots \\
\downarrow \mu & & \downarrow \mu & & \downarrow \mu & & \downarrow \mu & & \\
\cdots \to {}_sH_{q+1}(|K|, |K|-|L|) & \to & {}_sH_q(|K|-|L|) & \to & {}_sH_q(|K|) & \to & {}_sH_q(|K|, |K|-|L|) & \to & \cdots
\end{array}$$

in which the maps $\mu = j_* \nu$ (where $j\colon (|K \overset{*}{-} L|) \longrightarrow (|K|, |K|-|L|)$ is inclusion) are isomorphisms.

3. Simplicial manifolds.

A topological n-manifold is ordinarily defined to be a paracompact T_2 space such that every point has an open neighborhood homeomorphic to R^n. Similarily, we may define a pair of paracompact T_2 spaces (X, A) with A closed in X to be a relative n-manifold if $X - A$ is a topological n-manifold. The basic property of topological (or relative) manifolds that figures in duality is the following:

$$(3.1) \qquad H_i(X, X - \{x\}) = \begin{cases} Z & i = n \\ 0 & i \neq n \end{cases}$$

for all $x \in X - A$.

The proof, a fairly easy consequence of excision, is given in pp.293-4 of Spanier. We shall say that an arbitrary paracompact T_2 pair (X,A) with $A \subseteq X$ closed is a relative homology n-manifold if (3.1) holds. If R is a ring, R-homology manifolds may be defined similarly. Although it is not difficult to construct examples of homology n-manifolds that are not topological manifolds, we shall not do so here because the construction requires several long digressions.

We shall be interested in simplicial complex pairs (K,L) such that $(|K|,|L|)$ is a relative manifold. The following two lemmas are helpful in characterizing these objects via the simplicial structure. Henceforth all complexes will be assumed to be <u>finite</u>.

<u>Lemma 3.2</u>. Let K be a simplicial complex, $\alpha \in K$, and s_α the open simplex containing x. Then

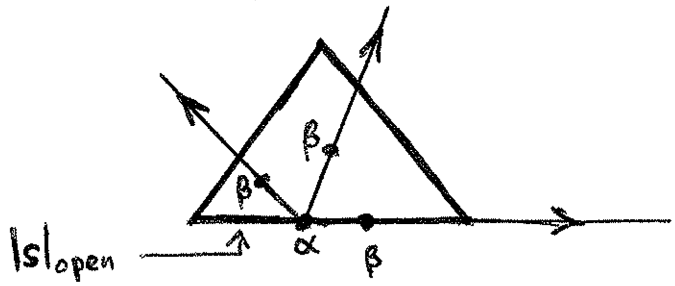
$$V_\alpha = |K| - \{\beta \in |K| \mid \beta \in |t|_{\text{open}} \text{ and } s_\alpha \subseteq t\}$$

is a strong deformation retract of $|K| - \{\alpha\}$.

<u>Proof</u>. Suppose β is a point in the open set

$$|\text{st } \delta| = \{\gamma \in |t|_{\text{open}} \text{ and } s \subseteq t\}.$$

Consider the ray originating at α and passing through β. Its intersection with the simplex t such that $\beta \in |t|_{\text{open}}$ is a closed segment, one of whose endpoints is α and whose other endpoint is a point of $|t| - |t|_{\text{open}}$ (See the picture below; a rigorous formulation may be given using some obvious properties of convex sets).



The deformation retraction and homotopy are defined on $|t|$ by pushing out from α along these segments. Since everything agrees on intersections, this gives a well-defined deformation retraction and homotopy on all of V_α . Details of the proof are left to reader as an exercise.

<u>Lemma 3.3</u>. If (K,L) is a simplicial pair and $x \neq y$ are cells in (K,L), then $M = c\ell x \cap st y$ is acyclic or empty.

<u>Proof</u>. Assume the intersection is nonempty. Then $y \subseteq z \in c\ell x$ implies $y \subseteq x$ and hence $y \subsetneq x$. Hence as an abstract cell complex M is a $(\rho(y) + 1)$-fold suspension of a $(\rho(x) - \rho(y) - \rho)$-simplex, and therefore it is acyclic.

We next give the desired characterization of simplicial complexes that are homology manifolds.

<u>Theorem 3.4</u>. Suppose (K,L) is an n-dimensional simplicial pair then the following are equivalent:

(i) $(|K|,|L|)$ is a homology n-manifold.

(ii) For each cell x in (K,L), $st_{(K,L)}x$ is (n)-cyclic.

(iii) For each cell $x^p \in (K,L)$, the complex $\text{Link}_{(K,L)} x^p = \cup\{s \mid s$ is a simplex in ${}^1K, s \subseteq$ some simplex having barycenter x^p as a vertex, but barycenter x^p is not a vertex of $s\}$ has the homology of S^{n-p-1}.

Remarks 1. If (K,L) satisfies any (hence all) of the above conditions, it is called a relative <u>simplicial n-manifold</u>.

2. If the subcomplex Link x^p is <u>combinatorially</u> equivalent to $\partial\Delta_{n-p}$ (i.e., the two complexes have simplicially isomorphic subdivisions) for every x^p, then (K,L) is called a relative <u>combinatorial n-manifold</u>. Every relative combinatorial n-manifold is a relative topological manifold (see Hudson [8],pp.20,24-26) but there exist absolute topological manifolds that are not homeomorphic to combinatorial manifolds (see Kirby and Siebenmann [9],pp.748-749). Furthermore, every differentiable manifold has a canonically given triangulation, unique up to combinatorial equivalence, as a combinatorial manifold (see Munkres [18],pp.82,103). The geometrical structure of combinatorial manifolds has been studied extensively; see Hudson [8], Rourke-Sanderson [20], Stallings [23] or Zeeman [27] for further information.

<u>Proof</u>. (i) $\Leftrightarrow$ (ii): This follows immediately from Lemma 3.2 and the exact sequence of the triple $(|K|, |K| - \{\alpha\}, |K| - |stS_\alpha|) = (|K|, |K| - \{\alpha\}, |K - sts_\alpha|)$.

(Added, February 1980)

Shortly after these notes were written, there was a major breakthrough in our understanding of the relationship between simplicial, combinatorial, and topological manifolds. By 3.1-3.4 we know that every complex homomorphic to a topological manifold is a simplicial manifold, and by the Hauptvermutung theorems of C. Papakyriakopoulos, E. Moise, and E. M. Brown, this complex is automatically a combinatorial manifold in dimension ≤ 3. However, we now know that in dimension ≥ 5 every combinatorial manifold admits a triangulation by a second complex which is not a combinatorial manifold; actually, it turns out that one need only prove this result for S^n itself, and there are infinitely many such complexes for each n. There are three basic ingredients in the proof: (1) The existence of 3-dimensional combinatorial manifolds H^3 with the homology of S^3 but nontrivial fundamental group. (2) The existence of a canonical triangulation for the suspension of a finite simplicial complex. (3) A proof that the double suspension of H^3 is homeomorphic to S^5 (hence the (k+2)-ple suspension is homeomorphic to S^{5+k} also). The last step is of course the deep one and is mainly due to R. Edwards (a triple suspension theorem and a double suspension theorem in many cases), with finishing touches due to J. Cannon (a double suspension theorem in all cases). The case $n = 4$ is open.

It is still not known whether every compact unbounded topological manifold can be triangulated as a simplicial complex. However, it is known that in dimensions ≥ 6 the answer reduces to statements about homology 3-spheres (see the work of D. Galewski and R. Stern, for example).

Further details may be found in the papers (and books) of those mentioned.

(ii) ⇔ (iii): By Lemma 3.3 the (-p-1)-fold suspension of stx^p is a simple complex -- call it X^+. It is immediate from the definitions that ${}^1X$ can be identified with the set of all simplices in ${}^1K$ that are contained in the link, and therefore

$$\tilde{H}_*(\text{Link } x^p) \cong \tilde{H}_*({}^1X) \underset{\text{by } 1.10}{\cong} \tilde{H}_*(X) =$$

$$H_*(X^+) \cong H_{*-p-1}(stx^p) = \begin{cases} Z & * = n \\ 0 & * \neq n. \end{cases}$$

Corollary 3.5. Conditions (ii) and (iii) are independent of the choice of triangulation.

The next result records some basic properties of simplicial n-manifolds.

Proposition 3.6. (i) A simplicial n-manifold is homogeneous of dimension n.

(ii) Every (n-1)-cell is a face of exact two n-cells.

(iii) If $K \supset L_1 \supseteq L$ and (K,L) is a simplicial n-manifold, so is (K,L_1).

Proof. (i) Since stx is (n)-cyclic, something in dimension n must lie above x.

(ii) Let x^{n-1} be a face of $x_1^n,\ldots,x_r^n$. The the integral n-cycles for stx^{n-1} are given by linear combinations

$$\sum_{i=2}^{r} a_i \left([x^{n-1}:x_1^n]x_i^n - [x^{n-1}:x_i^n]x_1^n \right)$$

for $a_i \in Z$ arbitrary. Thus $H_n(\mathrm{st}\,x^{n-1})$ is cyclic only if $r = 2$.

(iii) $\mathrm{st}_{(K,L_1)}x = \mathrm{st}_{(K,L)}x$, so the first is (n)-cyclic if the second is. Hence

$$n \leq \dim(K,L_1) \leq \dim(K,L) = n .$$

Definition. A simplicial n-manifold is orientable over R if there is an orientation α such that

$$\sum \alpha(x^n)x^n$$

(the sum being taken over all n-cells in (K,L)) is an n-cycle with R coefficients. If not, it is nonorientable. If (K,L) is Z-orientable, then the class of $\sum \alpha(x^n)x^n \neq 0$ in $H_n(K,L;Z)$; no non-zero multiple of it can be a boundary because $C_{n+1}(K,L;Z) = 0$.

By Proposition 3.6 (ii) every manifold is Z_2-orientable. On the other hand, RP^{2n} is nonorientable since $H_{2n}(RP^{2n};Z) = 0$. The following result describes the basic homological characterization of (Z)-orientability:

Theorem 3.7. (K,L) is Z-orientable if and only if there is a class $z \in H_n(K,L)$ such that for all $\alpha \in |K| - |L|$, $j_{\alpha *}z$

is a generator of ${}_sH_n(|K|,|K| - \{\alpha\} \cong {}_sH_n(|K|,|K|-|sts_\alpha|)$ $\cong H_n(sts_\alpha) = Z$ ($j_{\alpha *}s$ the inclusion map $|K| \to (|K|,|K| - \{\alpha\})$).

Proof. ($\Rightarrow$) Consider the cycle z represented by $\sum \alpha(x)x$. Clearly the image of z in $H_n(st_\alpha)$ is a cycle, and its representative has the form

$$\sum_{x \in sts_\alpha} \alpha(x)x \qquad (\beta = \text{reorientation}).$$

This cycle z' has infinite order for the same reasons z does ($C_{n+1}(st_\alpha) = 0$). Furthermore it is not divisible by an integer $m \neq \pm 1$; if it were, $z' = mw + dw'$ would imply $z' = mw$ (again using $C_{n+1} = 0$), which contradicts the fact that the coefficients in the expression for the class representing z' are ± 1. Thus $j_{\alpha *}z = z'$ must be a generator of $H_n(sts_\alpha) = Z$.

($\Leftarrow$) Suppose that $\sum n_x x$ represents the given cycle z. Then $n_x x$ represents the image of z in $H_n(stx)(= H_n(sts_\alpha)$, where a generator and $C_*(stx) = H_*(stx)$ (all chains are in dimension n), this means that $n_x = \pm 1$.

This theorem has two important consequences.

Corollary 3.8. If (K,L) is an orientable relative simplicial n-manifold and $L \subseteq L_1$, then (K,L_1) is an orientable relative simplicial n-manifold.

Proof. If $z \in H_n(K,L)$ is the class satisfying the condition of Theorem 3.7, then the image of z in $H_n(K,L_1)$ also does because the following diagram commutes for $x \in |K| - |L_1|$:

$$\begin{array}{ccc} {}_sH_*(|K|,|L| & \xrightarrow{\ j^*\ } & {}_sH_*(|K|,|L_1|) \\ & \searrow & \downarrow \\ & & {}_sH_*(|K|,|K| - \{x\}) \end{array}$$

Corollary 3.9. If (K,L) and (K',L') are simplicial n-manifolds with $(|K|,|L|)$ homeomorphic to $(|K'|,|L'|)$, then one is orientable if and only if the other is.

Proof. The orientability criterion of Theorem 3.7 clearly depends only on the topological structure of $(|K|,|L|)$.

Remark and Convention. Everything we have done generalizes naturally to arbitrary subrings of the rationals (e.g., R-homology manifolds, orientability), and accordingly all our results have natural analogs in this case. We shall not bother to state them explicity, however. If this is done, "coefficients" must be interpreted as signifying R-module coefficients. Furthermore, there are no theorems in the nonorientable case if $\frac{1}{2} \in R$ (since Z_2 is not an R-module then).

4. The duality theorems.

We finally have the machinery we need to prove the duality theorems.

The most important part of the proof of the Poincaré duality theorem appears in the next result. Let (K,L) be a simplicial manifold, $D(K,L)$ the n-fold suspension of the dual cell complex $(K,L)^*$. Define a carrier function $S\colon D(K,L) \to K \overset{*}{-} L$ by

$$S(x^*) = \{xy_1 \ldots y_r \mid x < y_1 \ldots < y_r \text{ in } (K,L)\},$$

where x^* is the cell of $D(K,L)$ corresponding to x. It is easy to check that S is a closed acyclic block decomposition taking vertices to vertices. Notice that $|S(x^*)|$ is just the set of points in the cosimplex x^*.

<u>Theorem 4.1</u>. If (K,L) is an orientable simplicial n-manifold, $K \overset{*}{-} L$ is a subdivision of $D(K,L)$. If (K,L) is nonorientable, $K \overset{*}{-} L$ is a Z_2-subdivision of $D(K,L)$. In either case the subdivision is carried by S.

<u>Proof</u>. In view of Theorem 1.9, it suffices to show that $D(K,L)$ is simple if (K,L) is orientable and Z_2-simple otherwise. But the condition $st_{(K,L)}x$ is n-cyclic is equivalent to the condition that $c\ell_{D(K,L)}x^*$ is 0-cyclic. If (K,L) is orientable, the orientation on the n-cells determines an augmentation in $D(K,L)$; in the nonorientable case we at least get a Z_2-augmentation. These augmentations satisfy the condition for a simple complex, and the augmentation clearly makes $(c\ell_{D(K,L)}x^*)^+$ acyclic .

Notice that, in the orientable case, the subdivision chain map depends on the choice of orientation for the n-cells very strongly, but it is unique once this choice is made. For the cosimplex of x^r has dimension n-r, and hence S does not increase dimensions.

Throughout the rest of this section, all subcomplexes are full subcomplexes.

Theorem 4.2. (Poincaré Duality Theorem).

Let (K,L) be a simplicial n-manifold.

If (K,L) is orientable, then for all coefficients G,

$$H^p(K,L;G) \cong H_{n-p}(K \overset{*}{-} L:G) \cong {}_sH_{n-p}(|K| - |L|;G) \quad \text{and}$$

$$H_p(K,L;G) \cong H^{n-p}(K \overset{*}{-} L;G) \cong {}_sH^{n-p}(|K| - |L|;G)$$

If (K,L) is nonorientable, then the analogous results hold if $G = Z_2$.

Corollary 4.3. If K is an absolute orientable simplicial manifold, then $H^p(K;G) \cong H_{n-p}(K;G)$, all coefficient groups G. If K is nonorientable the analogous result holds if $G = Z_2$.

Proof. We only prove the statement concerning orientable manifolds, since the other proof is similar. But by the universal coefficient theorem and the natural isomorphism $\mathrm{Hom}(Z^n,G) \cong \mathrm{Hom}(Z^n,Z) \otimes G$, it suffices to prove the first isomorphisms for $G = Z$. The isomorphism $H^p(K;L) \cong H_{n-p}(K \overset{*}{-} L)$

follows from Theorem 1 and the isomorphism $H_{n-p}(K \overset{*}{-} L) \cong {}_sH_{n-p}(|K|-|L|)$ follows from 2.7 .

In order to reconcile our results with Spanier's, we shall derive the standard description of the duality isomorphism.

<u>Theorem 4.4</u>. The duality isomorphism $\Delta: H^p(K) \cong H_{n-p}(K)$ is realizable for ring coefficients by the formula $\Delta u = u \cap z$ (cap product), where z is an indivisible n-cycle into which all n-simplices enter.

We use the term ring here to denote a commutative ring with unit. Let γ be the map $\gamma(u) = u \cap z$.

Proof. Consider the following diagram:

$$\begin{array}{ccc} C^p(K;R) & \xleftarrow{\ \omega_n\ } & C_{n-p}(D(K);R) \\ \big\downarrow{\scriptstyle \gamma_\#} & & \big\downarrow{\scriptstyle \pi} \\ C_{n-p}(K;R) & \xrightarrow{\ \eta\ } & C_{n-p}({}^1K;R) \end{array}$$

where ω_n is the chain (-n)-isomorphism, π is the duality isomorphism, η is the map in 1.12, and $\gamma_\#$ is a chain realization for γ. $\gamma_\#$ is further determined by the condition that the cap product chain map comes from the Alexander-Whitney diagonal given by the same ordering as that giving η.

Under all these assumptions, it is not hard to show that, given the same orientation of the n-simplices in the n-cycle (and hence the same augmentation of $D(K)$), $\eta \circ \gamma \circ \omega_n$ and π

are augmentation preserving chain raps, both of which are carried by the acyclic carrier S taking $(A_o \ldots A_q)^*$ to A_q^*. Hence they agree in homology. In other words, $\eta_* \circ \gamma = \pi_*$, where we now understand that π_* is the isomorphism $H^p(K;R) \to H_{n-p}({}^1K;R)$. The isomorphism in Corollary 3 is then given by $\tau_*\pi_*$, where τ is the inverse subdivision map. But by 1.12,

$$\tau_*\pi_* = \tau_*\eta_*\gamma = \tau_*\sigma_*\gamma = 1\gamma = \gamma .$$

<u>Addendum</u>. A similar cap product description of the duality isomorphism is valid if $L \neq \phi$ (compare [1] and [24]).

The isomorphisms of Theorem 4.2 have some evident naturality properties for inclusions of subcomplexes $L_1 \subseteq L_2$. We denote the long exact homology and cohomology sequences of a pair by HS and CS respectively, and a map from one of these long exact sequences to another is merely an approximate diagram that commutes up to sign.

<u>Proposition 4.5</u>. Suppose (K,L) is orientable. Then the following diagram commutes, where the horizontal maps are duality isomorphisms and $L_1 \subseteq L_2$:

$$\begin{array}{ccc} CS(K,L_2) & \xrightarrow{\cong} & {}_sHS(|K|,|K|-|L_1|) \\ \downarrow i_* & & \downarrow i_* \\ CS(K,L_1) & \xrightarrow{\cong} & {}_sHS(|K|,|K|-|L_1|) \end{array}$$

The proof is left to the reader as an exercise. (Hint: Use naturality on the chain complex level to get chain level short exact sequence maps, then show the maps in question are isomorphisms by 4.2, and the five lemma).

A particular consequence of Proposition 4.5 is the existance of a duality isomorphism

$$H^*(L) \longrightarrow {}_sH_*(|K|,|K|-|L|)$$

that is natural in L and induced by chain level maps. Thus the same technique used to prove 4.5 yields a relative duality isomorphism

(4.6) $$H^*(L_1,L_2) \longrightarrow {}_sH_*(|K|-|L_2|,|K|-|L_1|).$$

Again, this isomorphism is natural for inclusion of pairs.

5. Manifolds with boundary.

There are similar duality theorems for manifolds with boundary. In the topological case a manifold with boundary is a paracompact T_2 space for which every point has a neighborhood homeomorphic to R^n of $R^N_+ = \{x \in R^n, x_n \geq 0\}$; these two conditions are mutally exclusive by invariance of domain (see Munkres [18],p.4), and points for which the latter condition holds form an (n-1)-manifold without boundary called the boundary of the given manifold. The interior is the complement of the boundary. Thus if M is a manifold with boundary ∂M, then

(i) $(M, \partial M)$ is a relative n-manifold,

(ii) If $x \in \partial M$, then $H_*(M, M - \{x\}) = 0$ by excision (it is isomorphic to $H_*(R^n_+, R^n_+ - \{0\})$).

This motivates the next definition.

<u>Definition</u>. A paracompact T_2 pair $(X, \partial X)$ with ∂X closed in X is called a <u>homology n-manifold with boundary</u> if

(i) $(X, \partial X)$ is a relative homology n-manifold

(ii) ∂X is (an absolute) homology (n-1)-manifold

(iii) If $x \in \partial X$, then $H_*(X, X - \{x\}) = 0$.

The boundary is said to be regular (or <u>collared</u>) provided

(iv) $(X, \partial X)$ is homeomorphic to $(X \times \{0\} \cup \partial X \times \{1\})$ so that ∂X corresponds to $\partial X \times \{1\}$ by the "identity".

Although this condition is not needed to prove the duality theorems, we shall assume it here in order to shorten the necessary preliminaries. A result of M. Brown ([3]; see Connelly [4] for a short proof) states that (iv) holds for all topological manifolds with boundary.

<u>Definition</u>. A simplicial pair $(K, \partial K)$ is called a <u>simplicial n-manifold with (regular) boundary</u> provided:

(i) $(|K|,|\partial K|)$ is a homology n-manifold with (regular) boundary

(ii) $(K,\partial K)$ is a relative simplicial n-manifold

(iii) ∂K is a simplicial (n-1)-manifold.

We first prove some general facts about simplicial manifolds with boundary. Although they are not needed for the duality theorems, they are interesting in their own right.

Lemma 5.1. (i) If $(K,\partial K)$ is a simplicial n-manifold with boundary and x^p is a simplex in ∂K, then $(\mathrm{Link}_K x^p)^+$ (augmented complex) is acyclic.

(ii) Every (n-1)-simplex of ∂K is contained in one and only one n-simplex of K.

Proof. (i) This follows by the same sort of argument needed to prove (i) ⇒ (ii) ⇒ (iii) in Theorem 3.2 and condition (iii) in the definition of homology manifold with boundary. (Exercise: write out the details).

(ii) By the argument for (i) ⇒ (ii) in Theorem 3.2, modified as suggested in the last sentence, the complex $\mathrm{st}_K y^{n-1}$ is acyclic for each (n-1)-simplex y^{n-1} in ∂K. Since the complex is n-dimensional and has only on (n-1)-cell (namely, y), it has exactly one n-cell.

Proposition 5.2. If $(K,\partial K)$ is a simplicial n-manifold with boundary that is orientable, then ∂K is an orientable simplicial (n-1)-manifold. Furthermore, "fundamental cycles" $\sum \alpha(x)x$

may be chosen for $(K,\partial K)$ and ∂K so that $\partial_* z_{(K,\partial K)} = z_{\partial K} \in H_{n-1}(\partial K)$ (∂_* = connecting homomorphism).

<u>Proof</u>. $z_{(K,\partial K)}$ has a representative $c = \sum \alpha(x^n)x^n$ which may be viewed as a chain in $C_n(K)$. Since c is a cycle in $C_n(K,\partial K)$, the boundary dc is the image of a chain $c' \in C_{n-1}(\partial K)$. In fact, 5.1 (ii) shows that $c' = \sum \beta(y^{n-1})y^{n-1}$, where $\beta(y^{n-1}) = \pm 1$ for each (n-1)-simplex in ∂K; for there is a summand of ± 1 in the coefficient of y^{n-1} for each x^n such that $y^{n-1} \subseteq x^n$, but only one such x^n exists. Since $dc' = ddc = 0$, ∂K is orientable. The formula, $\partial_* z_{(K,\partial K)} = z_{\partial K}$ is immediate from $dc = c'$ and the definition of ∂_*.

Before proving the duality theorems for manifolds with boundary, we record two homotopy-theoretic facts needed in the course of the proof.

(5.3). If (X,A) is a pair of spaces with $A \subseteq X$ closed, then $X \times \{0\}$ is a strong deformation retract of $X \times \{0\} \cup A \times [0,1]$. (The proof is easy). Similarly for $[0,1)$.

(5.4). Suppose that K_0, K_1 are subcomplexes of $K_0 \cup K_1$ and $K_0 \cap K_1 = L$. Then the projection map

$$k\colon |K_0| \times \{0\} \cup |L| \times I \cup |K_1| \times \{1\} \longrightarrow |K_0| \cup |K_1|$$

is a homotopy equivalence.

<u>Proof of (5.4)</u>. By the homotopy extension property for subcomplexes (Spanier, Ch. 2,3), the identity map on

$|K_0| \times \{0\} \cup |L| \times I$ is homotopic rel $|L| \times \{1\}$ to a map ℓ satisfying $\ell(x,t) = (x,1)$ for $x \in |L|$. Extend ℓ to a self-map $\tilde{\ell}$ of domain (k) by defining $\tilde{\ell} \,|\, |K_1| \times \{1\} =$ identity. By construction $\tilde{\ell}$ is homotopic to the identity rel $|K_1| \times \{1\}$. Since k is a quotient may (everything is compact T_2), ℓ factors through k as (say) $\tilde{\ell} = jk$; of course, $jk \simeq id$. On the other hand, by construction the homotopy $\ell \simeq id$ may be taken as $h_s(x,t) = (x, 1-t+st)$ for $x \in |L|$, and from this formula it is clear that the homotopy $\tilde{\ell} \simeq id$ passes to a homotopy $kj \simeq 1$ rel $|K_1|$.

The duality theorems now follow with a minimum of extra difficulty.

Theorem 5.5. (Lefschetz Duality Theorem for Manifolds with Boundary). Suppose that $(K, \partial K)$ is a simplicial n-manifold $L \subseteq K$ is a subcomplex.

(i) If $(K, \partial K)$ is orientable, with regular boundary, then

$$H^p(K, \partial K \cup L) \cong {}_sH_{n-p}(|K| - |L|) \quad \text{for}$$

all coefficients and all values of p. If $(K, \partial K)$ is non-orientable, the corresponding result holds for Z_2 coefficients.

(ii) The duality isomorphism $\Delta: H^p(K, \partial K) \to H_{n-p}(K)$ is realizable for ring coefficients by the formula $\Delta u = u \cap z$, where $z \in H_n(K, \partial K)$ is an indivisible n-cycle into which all n-simplices center.

(iii). If $L_1 \subseteq L_2$ are subcomplexes of K, then

$$H^p(L_2, L_1 \cup (\partial K \cap L_2)) \cong {}_sH_{n-p}(|K|-|L_1|, |K|-|L_2|)$$

under the appropriate orientability assumptions for the coeffiecients considered.

<u>Proof</u>. (i) Consider the complex

$$K^{\#} = K \times \{0\} \cup \partial K \times [0,1]$$

with the product triangulation on $\partial K \times [0,1]$ (see Eilenberg and Steenrod, p.70) in which $\partial K \times \{0\}$ and $\partial K \times \{1\}$ are subcomplexes isomorphic to ∂K. Since the boundary is regular, $(K^{\#}, \partial K \times \{1\})$ is a simplicial n-manifold with boundary (its underlying space pair is homeomorphic to that of $(K, \partial K)$). By Corollary 2.6, $(K^{\#}, \partial K \times \{1\})$ is orientable if and only if $(K, \partial K)$ is.

By ordinary duality for $(K^{\#}, \partial K \times \{1\})$ and 3.8,

$$H^p(K^{\#}, \partial K \times \{1\} \cup L \times \{0\} \cup (L \cap \partial K) \times [0,1])$$
$$\cong {}_sH_{n-p}(|K^{\#}|-|\partial K \times \{1\}|-|\partial K \times \{1\}|-|L \times \{0\}|$$
$$-|(L \cap \partial K) \times [0,1]|).$$

By excision, the left hand side is isomorphic to

$$H^p(K_+ \cup N, \partial K \times \{1\} \cup L \cup (L \cap \partial K) \times I),$$

and if k maps the above pair to $(K_+, L \cap \partial K)$ by forgetting the I-coordinate, then K^* is an isomorphism on $H^*(K_+)$ by the closed interval version of 5.3, on $H^*(L \cup \partial K)$ by (5.4), and hence on relative groups by the five lemma. On the other hand,

$$|K^{\#}| - |\partial K \times \{1\}| - |L \times \{0\}| - |(L \cap \partial K) \times [0,1]|$$
$$= (|K|-|L|) \times \{0\} \cup (|\partial K|-|L|) \times [0,1],$$

and $|K| - |L|$ is a strong deformation retract of this space by (5.3).

(ii) This follows from the cap product realization in the unbounded case (see Theorem 4.4 and its addendum) together with some naturality formulae for the cap product (details are left to the reader as an exercise).

(iii) The isomorphism constructed in (i) is natural with respect to inclusions $L_1 \subseteq L_2$ and is realized on the chain complex level. Thus the indicated isomorphism exists by the same methods used to study (4.6).

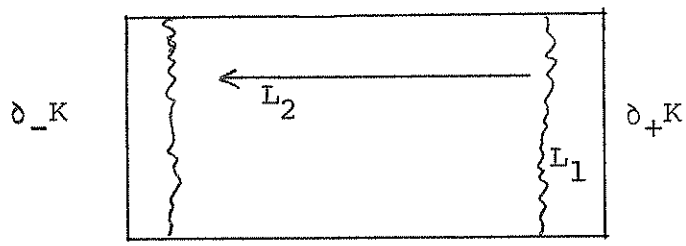
We shall conclude by giving a particularly important special case of 5.5 (iii).

<u>Proposition 5.6</u>. Suppose $(K,\partial K)$ is a simplicial n-manifold with regular boundary and ∂K is the disjoint union of $\partial_+ K$ and $\partial_- K$ (both simplicial (n-1)-manifolds). Then

$$H^p(K,\partial_+ K) \cong {}_sH_{n-p}(K,\partial_- K)$$

under the appropriate orientability assumptions for the coefficients considered.

<u>Proof</u>. Choose $L_2 = K \overset{*}{\text{———}} \partial_- K$, $L_1 = \overline{N^1(\partial_+ K)}$.



Thus 5.5 (iii) implies

$$H^p(L_2, L_1) \cong {}_sH_{n-p}(|K| - |L_1|, |K| - |L_2|).$$

Now $|\partial_+ K|$ is a deformation retract of $|L_1|$ by construction, and $|\partial_- K|$ is a deformation retract of $|K| - |L_2|$ since $|L_2| = |K| - |N^1(\partial_- K)|$ by 2. Thus we may write

$$H^p(L_2, \partial_+ K) \cong {}_sH_{n-p}(|K| - |L_1|, |\partial_- K|).$$

By 5.5 the inclusion $|K| - |L_1| \subseteq |K| - |\partial_+ K|$ induces an isomorphism in homology and cohomology and similarly for $|L_2| \subseteq |K| - |\partial_- K|$. Thus it suffices to prove that the inclusions $|K| - |\partial_+ K| \subseteq |K|$ induce isomorphisms in homology. We shall only consider the ∂_+ case; the ∂_- case follows from the same considerations.

Since the boundary is regular, we know that $|K|$ is homeomorphic to

$$|K| \times \{0\} \cup |\partial_+ K| \times [0,1] \cup |\partial_- K| \times [0,1]$$

By an easy extension of (5.3), the inclusions of $A = |K| \times \{0\} \cup |\partial_+ K| \times [0,1] \cup |\partial_- K| \times [0,1]$ in

$$B = |K| \times \{0\} \cup |\partial_+ K| \times [0,1] \cup |\partial_- K| \times [0,1]$$

and

$$C = |K| \times \{0\} \cup |\partial_+ K| \times [0,1] \cup |\partial_- K| \times [0,1]$$

are strong deformation retracts. Thus the pairs (B,A) and (C,A) have zero homology; by the long exact sequence of a triple, the same is true for (C,B). In other words $B \subseteq C$ induces an isomorphism in homology and cohomology, as was to be shown.

A slightly more delicate argument shows that Proposition 5.8 generalizes to the case $\partial_+ K \cap \partial_- K \neq \emptyset$ where $(\partial_\pm K, \partial_+ K \cap \partial_- K)$ are manifolds with (regular) boundary. This is left as an exercise for the reader.

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