A TOPOLOGICAL PROOF OF DE RHAM'S THEOREM

(Notes for MA 672, Fall 1975)

CONTENTS

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INTRODUCTION

Given a smooth manifold M (e.g., an open subset of some coordinate space), a fundamental construction assigns to M a cochain complex of differential forms. This object is defined more or less analytically in contrast to the singular cochain complex with (say) real coefficients, which is clearly topological in nature. A celebrated result due to G. de Rham states that these two cochain complexes have isomorphic cohomology, consequently providing a strong link between the topological and analytical properties of smooth manifolds. Our purpose is to give a proof of this **theorem** requiring a minimal amount of background information about differential manifolds. The proof given here is similar to one appearing in Hu's book [5], the main differences[1] being that a key feature left as an exercise in [5, Exercise 3D, p.167] is built into the proof, and the differential-geometric theorem of J.H.C. Whitehead on geodesically convex neighborhoods is replaced by the tubular neighborhood theorem, which is a more basic property from the topological point of view.

Besides Hu's book [5], other sources for proofs of de Rham's Theorem are Hirzebruch's book [4] and notes from a set of lectures by Bott [1] (the latter gives essentially the same proof as ours modulo a few appropriate elaborations). Another attempt at a short proof appears in [2]; however, the reader should be warned that this paper has several serious gaps.

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[1]Hu only formulates his proof for compact manifolds; however, his argument is easily adapted to the noncompact case.

Further material on the role of differential forms in topology may be found in a set of notes by E. Friedlander, P. Griffiths, and J. Morgan [X] (also see [10]).

1. Generalities on Differential Forms.

Suppose that V is an open subset of R^n and V is an open subset of R^m. We shall say that a function $f: U \to V$ is smooth if each coordinate function $f^1,\ldots,f^m$ has continuous partial derivatives of all orders. Such mappings are automatically continuous.

A differential k-form on an open subset U of R^n is an expression of the form

$$(1.1) \quad \sum_{i_1<\ldots<i_k} f_{i_1\ldots i_k} \, dx^{i_1}\wedge\ldots\wedge dx^{i_k} ,$$

where each $f_{i_1\ldots i_k}$ is a smooth function from U to R. The set of all k forms $\Lambda^k(U)$ is a real vector space, and if $h: V \to U$ is smooth, then a contravariantly functorial map $h^*: \Lambda^k(U) \to \Lambda^k(V)$ (the pullback) is induced. There are also wedge operations

$$\wedge: \quad \Lambda^k \otimes \Lambda^\ell \to \Lambda^{k+\ell}$$

and an exterior differentiation

$$d: \quad \Lambda^k \to \Lambda^{k+1}$$

The wedge product is characterized on "basic" forms $dx^{i_1}\wedge\ldots\wedge dx^{i_1}$ by the rule $dx^a \wedge dx^b = -dx^b \wedge dx^a$ (hence $dx^a \wedge dx^a = 0$) and associativity, while on more general forms it is characterized by $(f_1\omega_1)\wedge(f_2\omega_2) = f_1f_2(\omega_1 \wedge \omega_2)$

(we assume that f_1, f_2 are C^∞ = smooth functions), $1 \wedge \omega = \omega$, and distributivity. The exterior derivative is characterized by the rules $df = \sum \frac{\partial f}{\partial x^i} dx^i$,

$d(f\omega) = df \wedge \omega$, $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^r \omega_1 \wedge d\omega_2$ (where ω_1 is an r-form), and real-linearity. The pullback is characterized by the rules $h^*f = fh$, $h^*(\omega_1 \wedge \omega_2) = h^*\omega_1 \wedge h^*\omega_2$, $h^*d\omega = dh^*\omega$, and real-linearity. The reader who is so inclined should be able to use these to write out these operations for forms explicitly written as in (1.1) and obtain closed formulas. Using the explicit expression of (1.1) and the above rules, it is not difficult to derive the following basic formula:

(1.2) $d(d\omega) = 0$ for all forms ω.

In particular, the sequence

$$0 \to 0 \to \Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \dots \to \Lambda^n \to 0 \to \dots$$

is a cochain complex, and naturally our objective is to describe the cohomology of this cochain complex. For the time being we call it the de Rham cohomology and denote it by $H^*_{DR}(U)$.

Following standard terminology, we say a differential form is closed if $d\omega = 0$ and **exact** if $\omega = d\theta$ for some θ. Of course, $d(d\theta) = 0$ implies that exact forms are closed; the simplest example of a nonexact closed form is the standard object

$$\frac{x\,dy - y\,dx}{x^2+y^2}$$

defined on $R^2 - \{0\}$.

The necessary background for all this may be found in almost any element any treatment of differentiable manifolds; an especially direct and elementary approach appears in Rudin [8].

In conclusion, we state the appropriate "grade-commutative law" for wedge multiplication, leaving its proof as an (elementary but messy) exercise:

(1.3) If ω_1 is a p-form and ω_2 is a q-form, then

$$\omega_2 \wedge \omega_1 = (-1)^{pq}\, \omega_1 \wedge \omega_2 \ .$$

2. Smooth singular cohomology.

Let $\Delta_q \subseteq R^q$ be the q-simplex with vertices $0, e_1, \ldots, e_q$. Given an open subset U of R^n, a smooth singular q-simplex in U is defined to be a continuous map $T: \Delta_q \to U$ that extends to a smooth map on a neighborhood of Δ_q. Given this definition it is possible to define smooth singular chains, smooth singular homology, and smooth singular cohomology exactly as in the continuous case. Here are a few basic observations:

(2.1) The complex of smooth singular chains is a subcomplex of the ordinary singular complex.

(2.2) The smooth singular complex is functorial for smooth maps. Furthermore, the induced chain maps are just the restrictions of the usual maps on the ordinary chain complexes.

(2.3) If two smooth maps $f, g: U \to V$ are smoothly homotopic, the induced maps of smooth singular chains are chain homotopic. The chain homotopy is given by the restriction of the usual one.

(2.4) If $\mathcal{U}$ is an open covering of U, the inclusion of $\mathcal{U}$-small smooth singular chains induces an isomorphism in homology and cohomology.

The first two are obvious, and the last two may be proved using the corresponding arguments for ordinary singular homology and cohomology.

Notation. ${}_sH_*$ and ${}_sH^*$ denote the smooth singular homology and cohomology functors; we shall use real coefficients throughout these notes. Inclusion of subcomplexes induces natural transformations ${}_sH_* \to H_*$ and $H^* \to {}_sH^*$. Furthermore, if ${}_{\mathcal{U}}H$ or ${}_{\mathcal{U},s}H$ denotes the object formed from $\mathcal{U}$-small chains or cochains, the following squares are commutative:

$$\begin{array}{ccc} {}_{\mathcal{U},s}H_*(U) & \xrightarrow{\cong} & {}_sH_*(U) \\ \downarrow & & \downarrow \\ {}_{\mathcal{U}}H_*(U) & \xrightarrow{\cong} & H_*(U) \end{array} \qquad \begin{array}{ccc} H^*(U) & \xrightarrow{\cong} & {}_{\mathcal{U}}H^*(U) \\ \downarrow & & \downarrow \\ {}_sH^*(U) & \xrightarrow{\cong} & {}_{\mathcal{U},s}H^*(U) \end{array}$$

Of course, one can define relative groups ${}_sH_*(U,V)$ and ${}_sH^*(U,V)$ if V is an open subset of U, and they have the usual exactness properties. Furthermore, the ${}_sH_*$ long exact sequence maps naturally into the H_* long exact sequence, and similarly for $(H^*, {}_sH^*)$ replacing $({}_sH_*, H_*)$; the H and ${}_sH$ are switched because homology is covariant while cohomology is contravariant. Thus ${}_sH_*$ and ${}_sH^*$ satisfy all the Eilenberg-Steenrod axioms, which in turn strongly suggests the following result:

Theorem 2.5. The natural transformations ${}_sH_* \longrightarrow H_*$ and $H^* \longrightarrow {}_sH^*$ are isomorphisms.

Proof. We shall only do the proof for homology; this suffices because the corresponding cohomology transformation is formed by taking the vector space dual of the homology transformation by the Universal Coefficient Theorem.

Case 1. $U = R^0 = \{\text{point}\}$. In this case continuous singular simplices are the same as smooth ones, so the corresponding singular chain groups are equal.

Case 2. $U \subseteq R^n$ is a convex open subset. Let $x_0 \in U$, and define

$f: \{0\} \to U$ by $f(0) = x_0$, $g: U \to \{0\}$ by $g(x) = 0$, all $x \quad U$. Then gf = identity and $fg \simeq$ identity smoothly by the straight line homotopy $H(x,t) = tx_0 + (1-t)x$. Hence we have a commutative diagram

$$\begin{array}{ccc} {}_sH_*(\{0\}) & \longrightarrow & {}_sH_*(U) \\ \downarrow & & \downarrow \\ H_*(\{0\}) & \longrightarrow & H_*(U) \end{array}$$

in which all maps except perhaps the vertical right one are isomorphisms; of course, this implies that the remaining map is also an isomorphism.

<u>Case 3</u>. U _ R^n is a finite union of convex open sets. It follows from (2.1)-(2.4) and the comments about long exact sequences that there is an exact Mayer-Vietoris sequence in ${}_sH_*$ associated to the diagram

$$\begin{array}{ccc} U \cap V & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & U \cup V \end{array}$$

for U,V open in R^n, and this sequence maps into the corresponding one for ordinary singular homology.

Write $U = V_1 \cup \ldots \cup V_r$ with each V_i convex, and proceed by induction on r (the case $r=1$ is true by Case 2). Let $W = V_1 \cup \ldots \cup V_{r-1}$, so that $W \cap V_r = (V_1 \cap V_r) \cup \ldots \cup (V_{r-1} \cap V_r)$. By the induction hypothesis ${}_sH_*(W) \to H_*(W)$ is an isomorphism; on the other hand, each set $V_i \cap V_r$ is also convex, so that ${}_sH_*(W \cap V_r) \to H_*(W \cap V_r)$ is also an isomorphism by the induction hypothesis. Thus in the map of Mayer-Vietoris sequences associated to the diagram

$$\begin{array}{ccc} W \cap V_r & \longrightarrow & V_r \\ \downarrow & & \downarrow \\ W & \longrightarrow & U = W \cup V_r \end{array}$$

the homomorphisms ${}_sH_*(W \cap V_r) \to H_*(W \cap V_r)$ and

${}_sH_*(W) \oplus {}_sH_*(V_r) \to H_*(W) \oplus H_*(V_r)$ are isomorphisms, and therefore the homomorphism ${}_sH_*(U) \to H_*(U)$ is too by the five lemma.

<u>Case 4</u>. U is arbitrary. First of all, the map ${}_sH_*(U) \to H_*(U)$ is

onto. For if $z \in H_q(U)$, then there is a compact subset $K \subseteq U$ such that $z \in \text{Image } H_q(K)$ (If $z = \text{class of } \sum n_i T_i$, let $K = \cup T_i(\Delta_q)$). By compactness there are finitely many convex open sets $V_1, \ldots, V_r \subseteq U$ such that $K \subseteq V_1 \cup \ldots \cup V_r$, and it follows that z is the image of some class $z_0 \in H_q(V_1 \cup \ldots \cup V_r)$. But by Case 3 the top arrow in the following commutative diagram

$$\begin{array}{ccc} {}_sH_q(V_1 \cup \ldots \cup V_r) & \xrightarrow{\alpha} & H_q(V_1 \cup \ldots \cup V_r) \\ i_* \downarrow & & i_* \downarrow \\ {}_sH_q(U) & \xrightarrow{\alpha} & H_q(U) \end{array}$$

is an isomorphism, and it is immediate that $z = \alpha(i_* \alpha^{-1} z_0)$.

To conclude, we must show ${}_sH_*(U) \to H_*(U)$ is one-to-one. Suppose $z \in {}_sH_q(U)$ goes to zero in $H_q(U)$. By the same sort of argument used in the previous paragraph we may write

$z = i_* z_0$ for $z_0 \in {}_sH_q(V_1 \cup \ldots \cup V_r)$. By the above commutative diagram, $\alpha(z_0)$ goes to zero in $H_q(U)$; it follows that $\alpha(z_0)$ goes to zero in $H_q(V_1 \cup \ldots \cup V_r \cup W_1 \cup \ldots \cup W_s)$ for suitable open convex subsets $W_j \subseteq U$ (If $\alpha(z_0) = \text{class } d(\sum m_k S_k)$, take the W_j's to be a finite open covering of the compact set $\cup S_k(\Delta_{q+1})$). An inspection of the commutative diagram

$$\begin{array}{ccc} {}_sH_q(V_1 \cup \ldots \cup V_r) & \xrightarrow[\cong]{\alpha} & H_q(V_1 \cup \ldots \cup V_r) \\ h_* \downarrow & & \searrow \\ {}_sH_q(V_1 \cup \ldots \cup V_r \cup W_1 \cup \ldots \cup W_s) & \xrightarrow[\cong]{\alpha} & H_q(V_1 \cup \ldots \cup V_r \cup W_1 \cup \ldots \cup W_s) \\ \ell_* \downarrow & & \swarrow \\ {}_sH_q(U) & \xrightarrow{\alpha} & H_q(U) \end{array}$$

then shows that z_0 goes to zero in

${}_sH_q(V_1 \cup \ldots \cup V_r \cup W_1 \cup \ldots \cup W_s)$. But this means that

$z = \ell_* h_* z_0 = \ell_* 0 = 0$, proving that α is one-to-one.

3. Remarks on Presheaves.

We shall need a superficial acquaintance with a few simple sheaf-theoretic concepts. Given a topological space X, a presheaf on X is a contravariant functor from the category of open subsets of X (with inclusions as the morphisms) to some other category (real vector spaces in these notes). In plain English, given an open subset U of X, there is a real vector space $P(U)$, and if $V \subseteq U$ then there is a linear transformation $r_{uv}: P(U) \to P(V)$; these maps are assumed to satisfy

$$r_{uu} = \text{identity and } r_{vw}\, r_{uv} = r_{uw} \quad \text{if } W \subseteq V \subseteq U .$$

An obvious simple example is a constant presheaf; given a real vector space, S define $P(U) = S$ for all U and r_{uv} = identity. This is meant to suggest that presheaves might be suitable coefficients for an appropriate sort of cohomology. We shall not do this, but we shall prove some suggestive results in this direction; further details may be found in Spanier [9] or almost any book treating sheaf theory.

Suppose $\mathscr{U}$ is an open cover of X that is indexed by a subset of the positive integers. The q-dimensional Čech cochain group $C^q(\mathscr{U},P)$ is given by $\prod P(U_{i_0} \cap \ldots \cap U_{i_q})$, the product being taken over all

$(i_0,\ldots,i_q)$ with $i_0 < \ldots < i_q$ and $U_{i_0} \cap \ldots \cap U_{i_q} \neq \emptyset$. Given a q-dimensional cochain y, let $y_{i \ldots i_q}$ denote its component in the

$P(U_{i_0} \cap \ldots \cap U_{i_q})$-factor. The coboundary:

$$d: C^q(\mathscr{U},P) \to C^{q+1}(\mathscr{U},P)$$

is defined by the formula

$$(dy)_{i_0 \ldots i_{q+1}} = (-1)^q \sum_{j=0}^{q+1} (-1)^j\, r_j{}^*\, y_{i_0 \ldots \widehat{i_j} \ldots i_{q+1}}$$

where $r_j{}^*$ is the induced map from

$$P(U_{i_0} \cap \ldots \widehat{U_{i_j}} \ldots \cap U_{i_{q+1}}) \text{ to } P(U_{i_0} \cap \ldots \cap U_{i_{q+1}}).$$

A familiar calculation shows $d^2 = 0$, and the cohomology of the resulting cochain complex is denoted by $H^*(\mathcal{U},P)$ - the <u>Čech cohomology of the covering $\mathcal{U}$ with coefficients in</u> P.

A <u>morphism</u> of presheaves is a natural transformation from one presheaf (say P) to another (say Q); in other words, for each U there is a mapping $\theta_u: P(U) \to Q(U)$, and if $V \subseteq U$ the following diagram is commutative:

$$\begin{array}{ccc} P(U) & \xrightarrow{r_{uv}} & P(V) \\ \theta_u \downarrow & & \theta_v \downarrow \\ Q(U) & \xrightarrow{s_{uv}} & Q(V) \end{array}$$

<u>Examples</u>. Suppose X is an open subset of R^n.

1. Take $P(U)$ = all continuous real-valued functions on U, $r_{uv}: P(U) \to P(V)$ the restriction mapping.
2. Take $P(U)$ = all real-valued functions with (continuous) partial derivatives of all orders, r_{uv} the restriction mapping (again).
3. Take $P(U) = \Lambda^k(U)$, all differential k-forms, r_{uv} = restriction.
4. If P_1 and P_2 are the first two examples, then the inclusions of $P_2(U)$ in $P_1(U)$ for each U define a morphism of presheaves.
5. In example 3, exterior differentiation induces a presheaf morphism $d: \Lambda^k \to \Lambda^{k+1}$.

The following are easy to check:

<u>Proposition 3.1</u>. The class of all presheaves and presheaf morphisms on a given space X forms a category.

<u>Proposition 3.2</u>. Given a countably infinite open covering of X, the Čech cochain complexes and cohomology groups define functors on the category of presheaves.

<u>Definition</u>. A short exact sequence of presheaves is a diagram

$0 \to P \to Q \to R \to 0$

such that

$$0 \to P(U) \to Q(U) \to R(U) \to 0$$

is exact for each open subset $U \subseteq X$.

The following is also elementary to check:

Theorem 3.3. If $0 \to P \to Q \to R \to 0$ is a short exact sequence of pre-sheaves, then

$$0 \to C^q(\mathscr{U},P) \to C^q(\mathscr{U},Q) \to C^q(\mathscr{U},R) \to 0$$

is exact for each $q \geq 0$. Consequently, there is a long exact sequence

$$\cdots \to H^q(\mathscr{U},P) \to H^q(\mathscr{U},Q) \to H^q(\mathscr{U},R) \to H^{q+1}(\mathscr{U},P) \to \cdots$$

with $H^q = 0$ if $q < 0$.

Addendum to Theorem 3.3. If the diagram

$$\begin{array}{ccccccccc} 0 & \to & P & \to & Q & \to & R & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & P^1 & \to & Q^1 & \to & R^1 & \to & 0 \end{array}$$

is commutative with exact rows, then the following diagram also commutes:

$$\begin{array}{ccc} H^q(\mathscr{U},R) & \longrightarrow & H^{q+1}(\mathscr{U},P) \\ \downarrow & & \downarrow \\ H^q(\mathscr{U},R^1) & \longrightarrow & H^{q+1}(\mathscr{U},P^1). \end{array}$$

4. Proof of the Theorem For Open Sets.

The relation between differential forms and smooth singular chains and cochains is given by integration. Given a p-form ω on U and a smooth singular p-chain c on U, the integral $\int_c \omega$ is completely specified by the following properties:

$$\int_{c+c'} \omega = \int_c \omega + \int_{c'} \omega$$

$$\int_c \omega+\omega' = \int_c \omega + \int_c \omega'$$

$$\int_{rc} \omega = r\int_c \omega = \int_c r\omega \qquad \text{(r = a real number)}$$

$$\int_{f_* c} \omega = \int_c f^*\omega$$

$$\int_{I_q} h(x)dx^1\wedge\ldots dx^q = \int_{\Delta_q} h(x)d(x^1\ldots x^q) \ .$$

The last identity requires a little explanation. $I_q: \Delta_q \to U$ is the inclusion of Δ_q in some open neighborhood of itself, h is some smooth real-valued function defined on U, and the integral on the right is the usual Riemann or Lebesgue integral. If $q = 0$, the integral is interpreted to mean $h(0)$.

The integration operation gives a means of associating a smooth singular p-cochain to each differential form. Let

$$\theta^p_U: \Lambda^p(U) \to {}_sS^p(U)$$

be the associated mapping. The properties of the integral noted above imply that θ^p_U is a vector space homomorphism and is functorial in U. There is one additional important property that follows from the Generalized Stokes' Formula

$$\int_{dc} \omega = \int_c d\omega \ .$$

Namely, the following diagram is natural in U and commutative up to a sign factor:

$$\begin{array}{ccc} \Lambda^p(U) & \xrightarrow{\ d\ } & \Lambda^{p+1}(U) \\ \theta^p \downarrow & & \downarrow \theta^{p+1} \\ {}_sS^p(U) & \xrightarrow{\ \delta\ } & {}_sS^{p+1}(U) \end{array}$$

It follows that the θ-mappings send closed forms to cocycles and exact forms to coboundaries; this in turn implies that the θ^p mappings induce natural transformations

$$\theta^*: H^*_{DR} \to {}_sH^* .$$

Of course, the point of these notes is that θ^* is an isomorphism.

The starting point is the classical Poincaré Lemma, which may be stated as follows: Let $\underset{\sim}{R}$ denote the cochain complex with a copy of the reals in dimension zero and zeroes elsewhere, and include $\underset{\sim}{R}$ in $\wedge^*(U)$ via the constant maps (for each U). Assume that $U \subseteq R^n$ is a convex open set. Then the inclusion $\underset{\sim}{R} \to \wedge^*(U)$ induces an isomorphism in cohomology.

Proofs of this result may be found in books by Hu [5, pp.86-89] and Lang [7, pp.124-125]. On the other hand, the composite $\underset{\sim}{R} \to \wedge^*(U) \to {}_sS^*(U)$ also induces an isomorphism in cohomology; this is true because convex sets are smoothly contractible by "straight line" **homotopies** which in turn implies that ${}_sS^*(U)$ has a contracting chain homotopy. Summarizing, we have the following:

(4.1) θ^*_U is an isomorphism if U is a convex open subset of R^n .

In order to treat the general case, it will be necessary to use the following result:

Proposition 4.2. Let U be an open subset of R^n . Then U admits a countable open covering by convex open sets.

Proof. Since U is a subset of R^n , it is a separable metric space and therefore has the Lindelöf property. Apply this to the open covering of U consisting of all convex open subsets.

From now on we shall work with a fixed open covering of the type considered in Proposition 4.2. Consider the presheaves $\wedge^p$ and ${}_sS^p$ determined by differential forms and smooth singular chains on open subsets of U; the functoriality of θ^p implies that it induces a morphism of presheaves and thus induces transformations

$$\theta^{p\#}: C^*(\mathscr{U}, \Lambda^p) \to C^*(\mathscr{U}, {}_s S^p) .$$
$$\theta^{p*}: H^*(\mathscr{U}, \Lambda^p) \to H^*(\mathscr{U}, {}_s S^p) .$$

In fact, we can do some what more. Given $Y = \Lambda$ or ${}_{sin}S$, let $Z^p(Y)$ and $B^p(Y)$ be the sub-presheaves of cocycles and coboundaries in Y^p. Then there is the following commutative diagram of presheaves with exact rows:

$$(4.3) \qquad \begin{array}{ccccccccc} 0 & \to & Z^p(\Lambda) & \to & \Lambda^p & \to & B^{p+1}(\Lambda) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & Z^p({}_s S) & \to & {}_s S^p & \to & B^{p-1}({}_s S) & \to & 0 \end{array}$$

Similarly, if H^p_{DR} and ${}_s H^p$ are the presheaves determined by de Rham and smooth singular cohomology there is another commutative diagram of the same type:

$$(4.4) \qquad \begin{array}{ccccccccc} 0 & \to & B^p(\Lambda) & \to & Z^p(\Lambda) & \to & H^p_{DR} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B^p({}_s S) & \to & Z^p({}_s S) & \to & {}_s H^p & \to & 0 \end{array}$$

Of course, we can apply Theorem 3.3 and its addendum to both of these diagrams. We shall pursue this point extensively; the first step is given by the following result:

<u>Theorem 4.5</u>. Suppose that $i > 0$. Then $H^i(\mathscr{U}, \Lambda^p) = H^i(\mathscr{U}, {}_s S^p) = 0$.

<u>Proof</u>. First consider the Λ^p case. Define a map

$$L: C^i(\mathscr{U}, \Lambda^p) \to C^{i-1}(\mathscr{U}, \Lambda^p)$$

as follows: Let $\{f_j\}$ be a smooth partition of unity subordinate to $\mathscr{U}$, and set

$$(L(\omega))_{j_0 \ldots j_i} = \sum f_j (\omega)_{j\, j_o \ldots j_{i-1}}$$

If $D^i \omega = 0$ ($D^i: C^i \to C^{i+1}$ is the coboundary), then a straightforward computation gives $\omega = D^{i-1} L\,\omega$, proving $H^i(\mathscr{U},\Lambda^p) = 0$ (details of the calculation are written out in Hu [5, pp.158-159]).

In the ${}_sS^p$ case, define a map

$$M\colon C^i(\mathscr{U}, {}_sS^p) \to C^{i-1}(\mathscr{U}, {}_sS^p)$$

as follows: To define $(M\omega)_{j_o \ldots j_{i-1}}$, it suffices to give its value on every smooth singular $(i-1)$-simplex T in $U_{j_o} \cap \ldots \cap U_{j_{i-1}}$. Let $j_T \leq j_0$ be the least positive integer for which $T(\Delta_{i-1}) \subseteq U_{j_T}$ (such an integer exists because $T(\Delta_{i-1}) \subseteq U_{j_0}$). If $j_T = j_0$, set

$(M\omega)_{j_0 \cdots j_{i-1}}(T) = 0$; if $j_T < j_0$, set $(M\omega)_{j_0 \cdots j_{i-1}}(T)$

$= (\omega)_{j_T j_0 \cdots j_{i-1}}(T)$. Another straightforward computation (see Hu [5, pp.160-161]) shows that $D^i\omega = 0$ implies $\omega = D^{i-1}M\omega$.

We now use (4.3) and (4.4).

<u>Theorem 4.6</u>. In the following commutative diagrams, the horizontal maps are isomorphisms for all $p > 0$ and $i > 0$ ($i \geq 0$ for (4.8)):

$$\begin{array}{ccc} H^{i+1}(\mathscr{U}, Z^{p-1}(\Lambda)) & \leftarrow & H^i(\mathscr{U}, B^p(\Lambda)) \\ \downarrow & & \downarrow \\ H^{i+1}(\mathscr{U}, Z^{p-1}({}_sS)) & \leftarrow & H^i(\mathscr{U}, B^p({}_sS)) \end{array} \tag{4.7}$$

$$\begin{array}{ccc} H^i(\mathscr{U}, B^p(\Lambda)) & \to & H^i(\mathscr{U}, Z^p(\Lambda)) \\ \downarrow & & \downarrow \\ H^i(\mathscr{U}, B^p({}_sS)) & \to & H^i(\mathscr{U}, Z^p({}_sS)) \end{array} \tag{4.8}$$

<u>Proof</u>. The horizontal maps in (4.7) are the coboundary homomorphsims in the long exact sequence diagram given by (4.3). They are isomorphisms because the cohomology groups of Λ^p and ${}_sS^p$ vanish by Theorem 4.5.

To prove the horizontal maps in (4.8) are isomorphisms, it suffices to notice that $H^p_{DR}(U_{j_0} \cap \ldots \cap U_{j_i}) = 0 = {}_s H^p(U_{j_0} \cap \ldots \cap U_{j_i})$ by the argument proving (4.1). For this implies that $C^*(\mathcal{U}, {}_s H^p)$

$= C^*(\mathcal{U}, H^p_{DR}) = 0$; hence

$H^*(\mathcal{U}, {}_s H^p) = H^*(\mathcal{U}, H^p_{DR}) = 0$, and the conclusion follows from (4.4) and Theorem 3.3.

Corollary 4.9. For every $p > 0$ we have the following commutative diagram, in which the horizontal maps are isomorphisms:

$$\begin{array}{ccc} H^1(\mathcal{U}, Z^{p-1}(\Lambda)) & \to & H^p(\mathcal{U}, Z^0(\Lambda)) \\ \downarrow & & \downarrow \\ H^1(\mathcal{U}, Z^{p-1}({}_s S)) & \to & H^p(\mathcal{U}; Z^0({}_s S)). \end{array}$$

The proof of de Rham's theorem for open subsets of R^n is completed by the following result:

Theorem 4.10. The commutative diagram of Corollary 4.9 fits into the following larger commutative diagram, in which all horizontal maps are isomorphisms $(p > 0)$:

$$\begin{array}{ccccccccccc} & & & & H^p_{DR}(U) & \to & H^1(\mathcal{U}, Z^{p-1}(\Lambda)) & \to & H^p(\mathcal{U}, Z^0(\Lambda)) & \to & H^p(\mathcal{U}, \underset{\sim}{R}) \\ & \swarrow & & & \downarrow & & \downarrow & & \downarrow & & \searrow \\ {}_s H^p(U) & \to & \mathcal{U}, {}_s H^p(U) & & & \to & H^1(\mathcal{U}, Z^{p-1}({}_s S)) & \to & H^p(\mathcal{U}; Z^0({}_s S)) & \to & H^p(\mathcal{U}, \underset{\sim}{R}) \end{array}$$

Here $\underset{\sim}{R}$ denotes the constant presheaf.

Proof That 4.10 ⇒ De Rham's Theorem: If $p > 0$, then the long horizontal composites are isomorphisms as is the vertical map on the extreme right; hence it is immediate that θ^* is an isomorphism. On the other hand, zero-dimensional cocycles in both Λ^* and ${}_s S^*$ are characterized as maps $U \to R$ that are constant on each connected (equivalently, arc) component of U, and this correspondence is preserved by θ^0 (which is evaluation). Therefore $\theta^*: H^0_{DR}(U) \longrightarrow {}_s H^0(U)$ is also an isomorphism. Finally, use Theorem 2.5 to conclude this proof.

<u>Proof of 4.10</u>. The theorem is true for the center square by Corollary 4.9. To prove the theorem for the right hand square, it suffices to show that there is a commutative diagram as follows for each $U_i \in \mathscr{U}$:

$$\begin{array}{ccc} Z^0(\Lambda)(U_i) & \xrightarrow{\cong} & \\ \downarrow & & R(U) \\ Z^0({}_s\ S)(U_i) & \xrightarrow{\cong} & \end{array}$$

But this follows easily from the arcwise connectedness of each U_i (compare the proof of de Rham's Theorem given above).

To prove the theorem for the left hand square, first consider the following part of the long exact **sequence** diagram:

$$(4.11)\quad \begin{array}{ccccccc} H^0(\mathscr{U},\Lambda^{p-1}) & \to & H^0(\mathscr{U},B^p(\Lambda)) & \to & H^1(\mathscr{U};Z^p(\Lambda)) & \to & 0 = H^1(\mathscr{U},\Lambda^{p-1}) \\ \downarrow & & \downarrow & & \downarrow & & \\ H^0(\mathscr{U},{}_s\ S^{p-1}) & \to & H^0(\mathscr{U},B^p({}_s\ S)) & \to & H^1(\mathscr{U},Z^p({}_s\ S)) & \to & 0 = H^1(\mathscr{U},{}_s\ S^{p-1}) \end{array}$$

Let $C = \Lambda$ or ${}_s\ S$; then (4.8) and the definition of zero-dimensional cohomology yield the following identifications, all of which are natural in C:

$$(4.12)\quad \begin{array}{ccccccc} H^0(\mathscr{U},B^p(C)) & \overset{\cong}{\to} & H^0(\mathscr{U};Z^p(C)) & = & Z^0(\mathscr{U};Z^p(C)) & = & Z^p(C)(U) \\ \uparrow & & & & \uparrow & & \uparrow \\ H^0(\mathscr{U},C^{p-1}) & & \xrightarrow{\quad = \quad} & & Z^0(\mathscr{U},C^{p-1}) & = & C^{p-1}(U) \end{array}$$

The only complication occurs with $C = {}_s\ S$, in which case

$$C^{p-1}(U) = {}_{\mathscr{U},s}\ S^{p-1}(U) \quad \text{and} \quad Z^p(C)(U) = {}_{\mathscr{U},s}\ Z^p(U).$$

If one takes the cokernels of the vertical maps and uses the identifications given by (4.11) and $H^p = Z^p/\text{image } C^{p-1}$, the commutative left-hand square is obtained. Furthermore, the horizontal maps are commutative by (4.12), and it is easy to verify that the triangle on the left may also be added (${}_s\ H^* \cong {}_{\mathscr{U},s}\ H^*$ by (2.4)).

5. The General Case.

We begin by summarizing all the necessary properties of the category of smooth (second countable) manifolds and smooth mappings:

(5.1) It contains the category of open subsets of R^n (i.e., U is open in an arbitrary R^n, $n \geq 0$) and smooth mappings as defined in Section 1 as a full subcategory.

(5.2) The cochain complex functors Λ^* and ${}_sS^*$ extend to this category, and similarly for the natural chain transformation θ^*.

(5.3) Every smooth manifold M is a smooth retract of an open subset U in some R^n; i.e., there are smooth functions $i: M \to U$, $r: U \to M$ such that ri = identity.

Both (5.1) and (5.2) are fairly obvious from the usual definitions of smooth manifolds, smooth maps, and exterior forms on an arbitrary smooth manifold. (5.3) is a trivial consequence of the tubular neighborhood theorem applied to a smooth embedding of the manifold in some R^n (see Lang [7, pp.96-98]).

It will suffice to establish the following very general result:

Theorem 5.4. Let $\mathscr{C}$ be a category, let $\mathscr{C}_0 \subseteq \mathscr{C}$ be a full subcategory, and assume every object in $\mathscr{C}$ is a $\mathscr{C}$-retract of an object in $\mathscr{C}_0$. Suppose that E and F are contravariant functors defined on $\mathscr{C}$, $\theta: E \to F$ is a natural transformation, and $\theta_X: E(X) \to F(X)$ is an isomorphism if $X \in \mathscr{C}_0$. Then θ_X is an isomorphism for all $X \in \mathscr{C}$.

To prove the general case of de Rham's theorem, first apply Theorems 5.4 and 4.11 to $\theta^*: H^*_{DR} \to {}_sH^*$ and then apply 5.4 and 2.5 to $H^* \to {}_sH^*$.

Proof of 5.4. Given $X \in \mathscr{C}$, choose $Y \in \mathscr{C}_0$, $i: X \to Y$, $r: Y \to X$ so that ri = identity. Consider the following commutative diagram:

$$\begin{array}{ccccc} E(Y) & \xrightarrow{i^*} & E(X) & \xrightarrow{r^*} & E(Y) \\ \downarrow{\theta_Y} & & \downarrow{\theta_X} & & \downarrow{\theta_Y} \\ F(Y) & \xrightarrow{i^*} & F(X) & \xrightarrow{r^*} & F(Y) \end{array}$$

Since $i^* r^*$ = identity on $E(X)$ or $F(X)$, it follows that i^* is onto and r^* is one-to-one. To see that θ_X is one-to-one, notice that $\theta_X u = \theta_X v$ implies

$$\theta_Y r^* u = r^* \theta_X u = r^* \theta_X v = \theta_Y r^* v \ ;$$

but θ_Y is an isomorphism, so that $r^* u = r^* v$, which in turn implies $u = v$. To see that θ_X is onto, given $u \in F(X)$ write $u = i^* v$ (since i^* is onto); since θ_Y is an isomorphism $v = \theta_Y w$ for some w , and thus

$$u = i^* \theta_Y w = \theta_X i^* w \ .$$

6. Comparison of Cup and Wedge Products.

The rule $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 \pm \omega_1 \wedge d\omega_2$ implies that the wedge of two closed forms is closed and the wedge of a closed form with an exact one is exact. Consequently there are well-defined bilinear maps

$$\wedge\colon H^p_{DR}(M) \otimes H^q_{DR}(M) \to H^{p+q}_{DR}(M)$$

sending $[\omega] \otimes [\theta]$ to $[\omega \wedge \theta]$. It is immediate that this product makes $H^*_{DR}(M)$ into a functorial graded algebra. On the other hand, there is the cup product map

$$\cup\colon H^p(M) \otimes H^q(M) \to H^{p+q}(M) \ ,$$

and it is natural to ask whether these correspond under the isomorphism in de Rham's Theorem. The answer is that they do. We shall not give all the details of the argument; the references mentioned at appropriate points contain the omitted steps.

Lemma 6.1. Let A be a p-simplex in R^n with vertices $a_0,\ldots,a_p$, and let B be a q-simplex in R^m with vertices $b_0,\ldots,b_q$. Then there is a simplicial decomposition of $A \times B \subseteq R^n \times R^m$ such that every point lies on at least one $(p+q)$-simplex and an arbitrary $(p+q)$-simplex of the decomposition has vertices

$$(a_{i_0}, b_{j_0}), \ldots, (a_{i_{p+q}}, b_{j_{p+q}})$$

where $i_t \geq i_{t-1}$, $j_t \geq j_{t-1}$ and EXACTLY ONE of these inequalities IS STRICT for each t.

This is a special case of the construction appearing in Chapter II, Section 8, of Eilenberg and Steenrod [3].

We shall also need an explicit singular chain $X_{pq} \in S_{pq}(\Delta_p \times \Delta_q)$ defined on page 243 of S. MacLane, Homology [7]. This chain contains plus or minus each of the ordered affine $(p+q)$-simplices mentioned in Lemma 6.1, and the sign is that of $\det T_S$, where T_S is the unique affine map sending the vertices of $\Delta p+q$ monotonically to those of the $(p+q)$-simplex $S \subseteq \Delta_p \times \Delta_q$ (the assertion about signs requires a little work). The information given on X_{pq} readily yields the following result:

Lemma 6.2. Let f be a smooth real-valued function defined near $\Delta_p \times \Delta_q \subseteq R^p \times R^q$.

Then

$$\int_{\Delta_p \times \Delta_q} f(t)dt = \int_{X_{pq}} f(t)dt^1 \wedge \ldots \wedge dt^{p+q} ,$$

where the left-hand side is the Riemann or Lebesgue integral and the right-hand side is the differential forms integral.

Again returning to Homology [7], p.243, we see there is a natural chain transformation

$$\gamma: {}_sS_*(M) \otimes {}_sS_*(N) \to {}_sS_*(M\times N)$$

such that if M,N are neighborhoods of Δ_p, Δ_q in R^p, R^q and σ_p, σ_q are inclusions, then $\gamma(\sigma_p \otimes \sigma_q) = X_{pq}$. This yields the main step in comparing the cup and wedge products.

Lemma 6.3. The following diagram is commutative:

$$\begin{array}{ccc}
\Lambda^p(M) \otimes \Lambda^q(N) & \xrightarrow{\ \times\ } & \Lambda^{p+q}(M\times N) \\
\downarrow \theta\otimes\theta & & \downarrow \theta \\
{}_sS^p(M) \otimes {}_sS^q(N) & \longrightarrow & {}_sS^{p+q}(M\times N) \\
\downarrow \boxtimes & \swarrow (\gamma|S_p \otimes S_q)^* & \\
[{}_sS_p(M) \otimes {}_sS_q(N)]^* & &
\end{array}$$

Notation: The top map X is the external wedge $p_M^{\ *}\omega \wedge p_N^{\ *}\lambda$ **; in coordinates, if** $\omega = f(x)dx^1 \wedge\ldots dx^p$ **and** $\lambda = g(y)dy^1\wedge\ldots\wedge dy^q$ **,**

then $\omega \times \lambda = f(x)g(y)\, dx^1\wedge\ldots\wedge dx^p \wedge dy^1\wedge\ldots\wedge dy^q$.

Given $f: S_p^p \to R$, $q: S_q \to R$, the product

$f \boxtimes g: S_p \otimes S_q \to R$ is defined by

$f \boxtimes g(u \otimes v) = f(u)g(v)$ (compare MacLane [7, p.222 (1.3)]).

Proof of 6.3. By the naturality properties of the above transformations, it suffices to consider the case in which M,N are open neighborhoods of Δ_p , Δ_q in R^p , R^q and evaluate both composites applied to $\omega \otimes \lambda$ on $\sigma_p \otimes \sigma_q$. Assume ω and λ are given in coordinates as above; then the value of $\boxtimes \circ \theta \otimes \theta \circ (\omega \otimes \lambda)$ at $\sigma_p \otimes \sigma_q$ is

$$\int_{\sigma_p} \omega \int_{\sigma_q} \lambda = \int_{\Delta_p} f(x)dx \int_{\Delta_q} g(y)dy$$

$$= \int_{\Delta_p \times \Delta_q} f(x)g(y)d(x,y) ,$$

the latter by Fubini's Theorem (see Rudin's Real and Complex Analysis or any text on measure theory). By Lemma 6.2, the last integral is

$$\int_{X_{pq}} f(x)g(y)\, dx^1\wedge\ldots\wedge dy^q = \int_{X_{pq}} \omega\times\lambda$$

(by the above formula for $\omega \times \lambda$), and by definition the latter is

$(\gamma|\overline{S_p} \otimes S_q)^* \circ \theta \circ (\omega \otimes \lambda)$ evaluated at $\sigma_p \otimes \sigma_q$.

The next lema is almost as important as the previous one in proving the main result:

<u>Lemma 6.4</u>. Suppose that $s+t \neq p+q$ but $(s,t) \pm (p,q)$. Then

$(\gamma|S_s \otimes S_t)^* \theta (\omega \otimes \lambda) = 0$.

<u>Proof</u>. Again by naturality properties it suffices to consider the case M,N = neighborhoods of $\Delta s, \Delta t$ and to evaluate at $\sigma_s \otimes \sigma_t$. The hypothesis implies $s < p$ or $t < q$; since $\dim M = s$ and $\dim N = t$, it follows that $\Lambda^p(M) = 0$ or $\Lambda^q(N) = 0$.

The preceding lemmas give us a cochain level formula relating cup and wedge products.

<u>Proposition 6.5</u>. Let ψ: $S_*(M \times N) \to S_*(M) \otimes S_*(N)$ be the Alexander Whitney map. Then

$$\theta(\omega) \times \theta(\lambda) = \psi^* \gamma^* \theta(\omega \times \lambda) .$$

<u>Proof</u>. The left-hand side is obtained by taking

$\psi^* \Pi_{p,q} \quad \theta \otimes \theta(\omega \otimes \lambda)$, where

$$\Pi_{p,q}: \sum_{\substack{s+t \\ p+q}} {}_s \; S_s(M) \otimes_s \; S_t(N) \to {}_s \; S_p(M) \otimes_s \; S_q(N)$$

is the projection map. By 6.4 the composite

$$\gamma^* \theta^*(\omega \times \lambda) = \Pi_{p,q}{}^* \; (\gamma|S_p \otimes S_q)^* \theta^*(\omega \times \lambda) \text{ , and hence}$$

$$\psi^* \gamma^* \theta^*(\omega \times \lambda) = \theta(\omega) \times \theta(\lambda) \quad \text{by 6.3.}$$

We now have our main result.

<u>Theorem 6.6</u>. Let θ: $H^*_{DR} \to H^*$ be the isomorphism of de Rham's Theorem. Then

$$\theta([\omega] \wedge [\lambda]) = \theta[\omega] \wedge \theta[\lambda] .$$

<u>Proof</u>. Let $M = N$ in the above discussion, and let D: $M \to M \times M$ be

the diagonal. Applying D^* to the right hand side of 6.4, we get $\theta(\omega) \cup \theta(\lambda)$. Applying D^* to the left, we get $D^*\psi^*\gamma^*\theta(\omega\times\lambda)$. But $\gamma\psi$ is chain homotopic to the identity (e.g., see the discussion in MacLane [7]), and hence

$$\psi^*\gamma^*\theta(\omega\times\lambda) = \theta(\omega\times\lambda) + dz \quad \text{for some } z \text{ if } d\omega = d\lambda = 0 \,.$$

Hence
$$\begin{aligned}\theta(\omega) \cup \theta(\lambda) &= D^*(\theta(\omega\times\lambda) + \delta z)\\ &= \theta\, D^*(\omega\times\lambda) + \delta D^* z\\ &= \theta(\omega\times\lambda) + \delta D^* z \,,\end{aligned}$$

the latter since $\omega \wedge \lambda = D^*(\omega\times\lambda)$ (look at the definition of external wedge). Thus if $d\omega = d\lambda = 0$, then $\theta(\omega) \cup \theta(\lambda)$ and $\theta(\omega \wedge \lambda)$ determine the same cohomology class.

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