

Orientations and connected sums

Fact about $\mathbb{C}P^2$ (see Hatcher)

$$H^i(\mathbb{C}P^2; \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 2, 4 \\ 0 & \text{otherwise} \end{cases}$$

With respect to cup product, the standard generator in degree 0 (the cocycle $f: S_0(\mathbb{C}P^2) \rightarrow \mathbb{Z}$ taking each singular 0-simplex to $1 \in \mathbb{Z}$) is a two-sided unit, and if c generates H^2 , then c^2 generates H^4 . This is true for both generators since $(-c)^2 = c^2$.

Topologically, $\mathbb{C}P^2 = U \cup V$ where
 U has the homotopy type of S^2 [THICKENED $\mathbb{C}P^1$]
 V has the homotopy type of a point [COMPLEMENT OF $\mathbb{C}P^1$]
 $U \cap V$ has the homotopy type of S^3 .

One way to view this is to look at the inverse images of these sets with respect to the quotient map $S^5 \rightarrow \mathbb{C}P^2$. Let's

view S^5 as the unit sphere in \mathbb{C}^3 so we have the standard decomposition

$$S(\mathbb{C}^3) = S(\mathbb{C}) \times D(\mathbb{C}^2) \cup D(\mathbb{C}) \times S(\mathbb{C}^2)$$

$|z_1|^2 \geq |z_2|^2 + |z_3|^2$
 $|z_1|^2 \leq |z_2|^2 + |z_3|^2$

↑
 inverse image V
 is the thickening of
 this to the set of all
 points with $|z_1| \neq 0$

↑
 inverse image U .
 is the thickening of
 this to the set of
 all points with
 $|z_2|^2 + |z_3|^2 \neq 0$.

Hence V is just homeo to $\mathbb{C}^2 = \mathbb{R}^4$,
 and $V \cap U = \{pt.\}$
 $= \mathbb{C}P^2 - U$

inverse image of $U \cap V =$ all points with $z_1 \neq 0$ $(z_2, z_3) \neq (0, 0)$.

Suppose we try to form connected sums using the different possible identifications
 In principle we have the following:

$\mathbb{C}P^2 \#_{\varepsilon} \mathbb{C}P^2$ is formed from

$$U_1 \amalg U_2$$

disjoint union of two copies of U

with an identification of

$$U_1 \cap V_1 \cong S^3 \times \mathbb{R} \cong \mathbb{C}^2 - \{0\}$$

and $V_1 \cap V_2 \cong \text{same}$ by a

diffeomorphism $h \times \text{id}_{\mathbb{R}}$ on $S^3 \times \mathbb{R}$ which is orientation preserving or reversing
 $\varepsilon = -1$ $\varepsilon = +1$.

Check what happens in cohomology:

$$H^i(\mathbb{C}P^2 \#_{\varepsilon} \mathbb{C}P^2) \cong \begin{cases} \mathbb{Z} & i=0, 4 \\ \mathbb{Z} \oplus \mathbb{Z} & i=2 \\ 0 & \text{otherwise.} \end{cases}$$

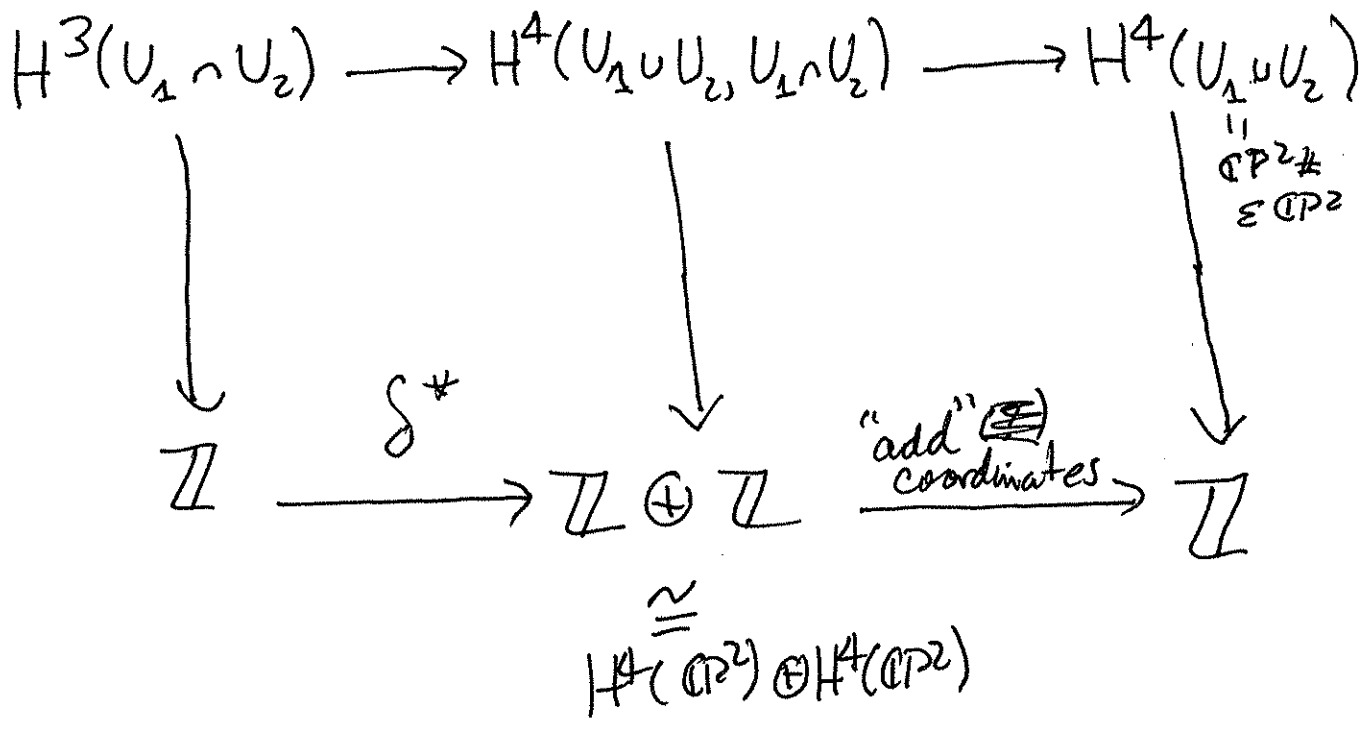
H^2 generated by α, β .
 α, β .

How do ~~$\alpha + \beta$~~ behave with respect to the cup product?
 \neq

We identify $U_1 + V_2$ with their images in $\mathbb{CP}^2 \# \mathbb{CP}^2$. Also note $U_i - V_i \cong \mathbb{CP}^1 = S^2$. Let W correspond to $U_1 \cap V_1 \xrightarrow{\cong} U_1 \cap U_2 \cong U_2 \cap V_2$ (they are identified).

Consider the relative Mayer-Vietoris sequence for $(U_1 \cup U_2; U_1, U_2)$. By excision we have $H^*(U_1 \cup U_2, U_2) \cong H^*(U_1, U_1 \cap U_2)$.
 $= H^*(U_1, U_1 \cap V_1) \cong H^*(\mathbb{CP}^2, V) \cong \begin{cases} H^i(\mathbb{CP}^2) & i > 0 \\ 0 & i = 0. \end{cases}$

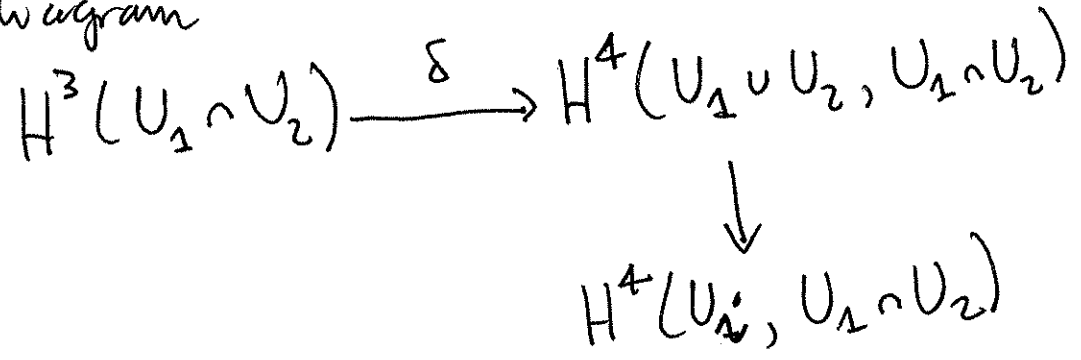
We have $H^*(U_1 \cup U_2, U_1 \cap U_2) \cong H^*(U_1, U_1 \cap U_2) \oplus H^*(U_2, U_1 \cap U_2)$ which is isomorphic to $H^*(\mathbb{CP}^2) \oplus H^*(\mathbb{CP}^2)$ in positive dimensions. This means that the behavior of the 2-dimensional cup product is given as follows:



~~CP^2~~
 α^2 generates the 1st summand
 β^2 generates the 2nd one.

CLAIM: The projection of δ^* onto either factor is an isomorphism. — Look at the

diagram



and notice that $H^3(U_i) \cong H^3(S^2) = 0$,
and likewise $H^4(U_i) = 0$.

So $\delta^*(\text{generator}) = (\eta_1 \text{ generator}, \eta_2 \text{ generator})$
where η_1, η_2 are ± 1 . This implies that
in $H^4(U_1 \cup U_2)$ we have $\alpha^2 = -\eta_1 \eta_2 \beta^2$.

If we change the gluing map on $U_1 \cap U_2 \cong S^3 \times \mathbb{R}$ by composing it with an orientation reversing diffeo of S^3 , then in the Mayer-Vietoris sequence the map δ^* will change; specifically, if $\delta_{\text{OLD}}^*(1) = (\eta_1, \eta_2)$, then $\delta_{\text{NEW}}^*(1) = \pm (\eta_1, -\eta_2)$. So with one choice of gluing map we get $\alpha^2 = \beta^2$ and with the other we get $\alpha^2 = -\beta^2$. In the two cases we get different conclusions about the cup product structure for $\mathbb{C}P^2 \# \mathbb{C}P^2$.

[A] In one case, every cup square is a positive integral multiple of a suitably chosen generator of H^4

[B] For each generator of H^4 there are classes y_{\pm} such that $(y_{\pm})^2$ is \pm that generator.

Since homotopy equivalences preserve cup product structure, it follows that the two possibilities yield homotopically inequivalent spaces!