Orientations and connected sums

Facts about $\mathbb{CP}^2$ (see Hatcher)

$$H^i(\mathbb{CP}^2;\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0,2,4 \\ 0 & \text{otherwise} \end{cases}$$

With respect to cup product, the standard generator in degree 0 ($\leftrightarrow$ the cocycle $f: S_0(\mathbb{CP}^2)\to\mathbb{Z}$ taking each singular 0-simplex to $1\in\mathbb{Z}$) is a two sided unit, and if c generates H^2, then c^2 generates H^4. This is true for ___both generators___ since $(-c)^2=c^2$.

Topologically, $\mathbb{CP}^2 = U\cup V$ where
U has the homotopy type of S^2 [THICKENED $\mathbb{CP}1$]
V has the homotopy type of a point [COMPLEMENT OF $\mathbb{CP}1$]
$U\cap V$ has the homotopy type of S^3.

One way to view this is to look at the inverse images of these sets with respect to the quotient map $S^5 \longrightarrow \mathbb{CP}^2$. Let's view S^5 as the unit sphere in $\mathbb{C}^3$ so we have the standard decomposition

unit sphere
$$S(\mathbb{C}^3) = \underset{|z_1|^2 \geq |z_2|^2+|z_3|^2}{S(\mathbb{C})\times D(\mathbb{C}^2)} \cup \underset{|z_1|^2 \leq |z_2|^2+|z_3|^2}{D(\mathbb{C})\times S(\mathbb{C}^2)}$$

inverse image V is the thickening of this to the set of all points with $|z_1| \neq 0$

inverse image U is the thickening of this to the set of all points with $|z_2|^2+|z_3|^2 \neq 0$.

Hence V is just homeo to $\mathbb{C}^2 = \mathbb{R}^4$, and $V \setminus U = \{pt.\} = \mathbb{C}P^2 - U$

inverse image of $U \cap V$ = all points with $z_1 \neq 0$ $(z_2, z_3) \neq (0,0)$.

Suppose we try to form connected sums using the different possible identifications

In principle we have the following:

$\mathbb{C}P^2 \#_\varepsilon \mathbb{C}P^2$ is formed from

$U_1 \coprod U_2$ disjoint union of two copies of U

with an identification of

$U_1 \cap V_1 \cong S^3 \times \mathbb{R} \cong \mathbb{C}^2 - \{0\}$

and $V_2 \cap V_2 \cong$ same by a diffeomorphism $h \times id_{\mathbb{R}}$ on $S^3 \times \mathbb{R}$ which is orientation preserving or reversing
$\varepsilon = -1$ $\varepsilon = +1$.

Check what happens in cohomology:

$$H^i(\mathbb{C}P^2 \#_\varepsilon \mathbb{C}P^2) \cong \begin{cases} \mathbb{Z} & i = 0, 4 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

H^2 generated by α, β.

How do ~~…~~ $\alpha + \beta$ behave with respect to the cup product? α, β.

We identify U_1 + V_2 with their images in $\mathbb{C}P^2 \# \varepsilon\, \mathbb{C}P^2$. Also note $U_i - V_i \cong \mathbb{C}P^1 = S^2$. Let W correspond to $U_1 \cap V_1 \sim U_2 \cap V_2$ ($\nearrow U_1 \cap U_2$) (they are identified).

Consider the relative Mayer-Vietoris sequence for $(U_1 \cup U_2; U_1, U_2)$. By excision we have $H^*(U_1 \cup U_2, U_2) \cong H^*(U_1, U_1 \cap U_2)$

$$= H^*(U_1, U_1 \cap V_1) \cong H^*(\mathbb{C}P^2, V) \cong \begin{cases} H^i(\mathbb{C}P^2) & i > 0 \\ 0 & i = 0. \end{cases}$$

We have $H^*(U_1 \cup U_2, U_1 \cap U_2) \cong$ $H^*(U_1, U_1 \cap U_2) \oplus H^*(U_2, U_1 \cap U_2)$ which is isomorphic to $H^*(\mathbb{C}P^2) \oplus H^*(\mathbb{C}P^2)$ in positive dimensions. This means that the behavior of the 2-dimensional cup product is given as follows:

$$\begin{tikzcd}
H^3(U_1 \cap U_2) \arrow[r] \arrow[d] & H^4(U_1 \cup U_2, U_1 \cap U_2) \arrow[r] \arrow[d] & H^4(U_1 \cup U_2) \arrow[d] \\
\mathbb{Z} \arrow[r, "\delta^*"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "\text{``add'' coordinates}"] & \mathbb{Z}
\end{tikzcd}$$

$U_1 \cup U_2 = \mathbb{CP}^2 \# \varepsilon \mathbb{CP}^2$

$\mathbb{Z} \oplus \mathbb{Z} \cong H^4(\mathbb{CP}^2) \oplus H^4(\mathbb{CP}^2)$

α^2 generates the 1st summand
β^2 generates the 2nd one.

<u>CLAIM</u>: The projection of δ^* onto either factor is an isomorphism. — Look at the diagram

$$\begin{tikzcd}
H^3(U_1 \cap U_2) \arrow[r, "\delta"] & H^4(U_1 \cup U_2, U_1 \cap U_2) \arrow[d] \\
& H^4(U_i, U_1 \cap U_2)
\end{tikzcd}$$

and notice that $H^3(U_i) \cong H^3(S^2) = 0$, and likewise $H^4(U_i) = 0$.

So $\delta^*(\text{generator}) = (\eta_1 \text{ generator}; \eta_2 \text{ generator})$ where η_1 & η_2 are ± 1. This implies that in $H^4(U_1 \cup U_2)$ we have $\alpha^2 = -\eta_1\eta_2\beta^2$.

If we change the gluing map on $U_1 \cap U_2 \cong S^3 \times \mathbb{R}$ by composing it with an orientation reversing diffeo of S^3, then in the Mayer-Vietoris sequence the map δ^* will change; specifically, if $\delta^*_{OLD}(1) = (\eta_1, \eta_2)$, then $\delta^*_{NEW}(1) = \pm(\eta_1, -\eta_2)$. So with one choice of gluing map we get $\alpha^2 = \beta^2$ and with the other we get $\alpha^2 = -\beta^2$. In the two cases we get different conclusions about the cup product structure ~~of~~ for $\mathbb{C}P^2 \# \varepsilon\, \mathbb{C}P^2$.

$\boxed{A}$ In one case, every cup square is a positive integral multiple of a suitably chosen generator of H^4

$\boxed{B}$ For each generator of H^4 there are classes $y_\pm$ such that $(y_\pm)^2$ is $\pm$ that generator.

Since homotopy equivalences preserve cup product structure, it follows that the two possibilities yield homotopically inequivalent spaces!