Cutting and pasting

We begin by quoting the following description appearing on page 55 of C. T. C. Wall's 1960 – 1961 Differential Topology notes, which available are online at **http://www.maths.ed.ac.uk/~aar/surgery/wall.pdf**.

> Cutting and gluing are simple geometrical constructions which, given some smooth manifolds (possibly with boundaries or corners) and additional data where necessary, give rise to new manifolds. On account of their perspicuity, these methods were much used in the days of topology of surfaces, and they remain a very powerful tool [***Note:*** Although this was written nearly 50 years ago, it is still true today!].

Bicollared submanifolds. Suppose that M is a connected smooth n – manifold without boundary and N is a connected smooth $(n-1)$ – dimensional submanifold without boundary such that N is a closed subset of M. We shall say that N is bicollared in M provided N has a tubular neighborhood given by a trivial 1 – dimensional vector bundle $N \times \mathbb{R} \to M$.

One can completely determine whether a smooth submanifold N as above is bicollared by means of invariants called ***characteristic classes*** (more precisely, by the ***first Stiefel – Whitney class***). The standard reference for this topic is the following classic book:

> **J. W. Milnor and J. D. Stasheff.** ***Characteristic classes*** (Annals of Math. Studies No. 76). Princeton University Press, Princeton, NJ, 1974.

In particular, if M and N are both orientable, then it follows that N is bicollared in M.

Example. The smoothness condition is absolutely necessary. In particular, there are "wild" topological embeddings of S^2 in $\mathbb{R}^3$ which are not even topologically bicollared. Probably one of the most famous is the ***horned sphere*** constructed by J. W. Alexander in 1924. This example is discussed on pages 170 – 172 of Hatcher, and there are also some curious **YouTube** videos like the following:

http://www.youtube.com/watch?v=d1Vjsm9pQlc

http://www.youtube.com/watch?v=Pe2mnrLUYFU&NR=1

We are particularly interested in the case where N is ***two – sided*** in the sense that the complement $M \setminus N$ consists of two components; for example, the standard inclusion of S^n in $\mathbb{R}^{n+1}$ is two – sided but the slice inclusion of S^1 as $S^1 \times \{1\}$

in $S^1 \times S^1$ is not (although it is locally two – sided). Before proceeding, we note one important feature of this example which generalizes and an elementary result showing that a bicollared submanifold is either one – sided or two – sided:

<u>Exercise 1.</u> In the setting above, suppose that N is bicollared but not two – sided. Then there is a surjection from $\pi_1(M)$ to $\mathbb{Z}$. [***<u>Hint:</u>*** Let B be a closed tubular neighborhood of N, so that B is diffeomorphic to $N \times [-1, 1]$, and let P be a "vertical" line segment joining $(x, -1)$ to $(x, 1)$ for some x in N. Next, let C be the closure of the complement of B and construct an embedded curve Q in C joining the same two points. The union of P and Q then defines a closed curve from S^1 to M; use the Tietze Extension Theorem to extend the identity on Q to a map $C \to Q$, and note that projection onto $[-1, 1]$ may be interpreted as an extension of the identity on P to a map $N \to P$. Explain why it follows that the curve S^1 to M determines a retract up to homotopy.]

<u>Exercise 2.</u> Suppose that X is a locally arcwise connected space and A is a closed nowhere dense subset of X, and assume further that A has an open neighborhood U such that $U \setminus A$ has k (arcwise) connected components. Prove that $X \setminus A$ has at most k components. Using this result, explain why the complement of a bicollared submanifold has at most two components. [***<u>Hint:</u>*** If W is an arc component of $U \setminus A$ explain why W must have a limit point which lies in A.]

Note that if M is simply connected, then there is no surjection from $\pi_1(M)$ to $\mathbb{Z}$, and therefore if N is bicollared in such a manifold then it is two – sided.

We now have the following theorem on ***cutting*** a smooth manifold into two manifolds with boundary.

<u>Theorem.</u> Suppose that N is a connected, bicollared, two – sided smooth submanifold of the connected manifold M, and assume that N is also a closed subset of M. Let U and V be the connected components of $M \setminus N$. Then the closures $\overline{U}$ and $\overline{V}$ are smooth submanifolds with boundary $N = \partial \overline{U} = \partial \overline{V}$.

<u>All</u> of the preceding results have analogs for topological manifolds.

Pasting constructions

There is also a reverse process of ***pasting*** or ***gluing*** together two manifolds with boundary by means of a homeomorphism or diffeomorphism between the boundaries. The first step in describing this construction is a simple but important criterion for recognizing when certain quotient spaces are Hausdorff.

Proposition 1. Suppose that X and Y are Hausdorff topological spaces and that $h: U \to V$ is a homeomorphism from an open subset of X to an open subset of Y. Assume further that we have the following separation criterion:

> There are open neighborhoods W_1 and W_2 of $X \setminus U$ and $Y \setminus V$ in X and Y respectively such that $h[W_1 \cap U]$ and $W_2 \cap V$ are disjoint.

Then there is a **Hausdorff** space P and there are open embeddings $f: X \to P$ and $g: Y \to P$ such that $f|U$ is the composite $(g|V)\,h$, and if $\varphi: X \to Q$, $\psi: Y \to Q$ are two continuous mappings such that $\varphi|U$ is the composite $(\psi|V)\,h$, then there is a unique continuous mapping $\Lambda: P \to Q$ such that $\Lambda f = \varphi$ and $\Lambda g = \psi$.

Standard arguments imply that the conditions in the proposition characterize the space P uniquely up to homeomorphism. It is somewhat predictable that one forms P by taking the disjoint union of X and Y, and then factoring out the equivalence relation associated to the homeomorphism h. The main challenge is verifying that P is indeed Hausdorff; obviously, the idea is to use the separation crieterion. The usual **counterexample** yielding a non – Hausdorff space when the separation criterion is not valid has $X = Y = \mathbb{R}$ and $U = V = \mathbb{R} \setminus \{0\}$, with h equal to the identity map (the two copies of $\mathbf{0}$ do not have disjoint neighborhoods).

The next step in describing this construction is the following elementary result on gluing together a pair of manifolds using an identification of an open subset of one with an open subset of the other (this can be done using techniques from Mathematics 205C).

Proposition 2. Suppose that M and N are k – manifolds without boundary and that $h: U \to V$ is a diffeomorphism from an open subset of M to an open subset of N. Assume also that the separation criterion in the preceding result is valid. Then there is a smooth k – manifold P and there are smooth embeddings $f: M \to P$ and $g: N \to P$ such that $f|U$ is the composite $(g|V)\,h$, and if $\varphi: M \to Q$, $\psi: N \to Q$ are two smooth mappings such that $\varphi|U$ is the composite $(\psi|V)\,h$, then there is a unique smooth mapping $\Lambda: P \to Q$ such that $\Lambda f = \varphi$ and $\Lambda g = \psi$.

As in the first proposition, standard arguments imply that the conditions in the proposition characterize the manifold P uniquely up to diffeomorphism.

Our pasting or gluing construction will use the preceding result plus the existence and uniqueness of collar neighborhoods for boundaries.

Definition. Let M and N be topological or smooth manifolds with boundaries, and suppose we are given a homeomorphism/diffeomorphism $h: \partial M \to \partial N$. Let c_M and c_N be closed collar neighbhorhood embeddings from $\partial M \times [0, 1]$ and $\partial N \times [0, 1]$ to M and N respectively, and call their images U and V respectively. Then $M \cup_h N$ is the manifold formed from $\mathrm{INT}(M)$ and $\mathrm{INT}(N)$ by gluing U to V along the homeomorphism/diffeomorphism H defined by

$$H(c_M(x, t)) = c_N(h(x), 1 - t).$$

One can check directly that the separation criterion of Proposition **1** is valid for this sort of example (choose sufficiently thin collars).

Proposition. The manifold $\mathcal{A} = M \cup_h N$ is homeomorphic to the quotient space of the disjoint union $M \coprod N$ by identifying x in ∂M with $h(x)$ in ∂N. In the smooth category, $\mathcal{A}$ contains smooth submanifolds M' and N' diffeomorphic to M and N such that $M' \cap N'$ is their common boundary.

Proof. Take M' and N' to be $M \setminus c_M[\partial M \times [0, ½)]$ and $N \setminus c_N[\partial N \times [0, ½)]$ respectively. Topologically $\mathcal{A}$ is the union of M' and N' such that the intersection is the common boundary, which is $c_M[\partial M \times \{½\}] = c_N[\partial N \times \{½\}]$. To see that M' and N' are diffeomorphic to M and N in the smooth category, first notice that one can construct an increasing diffeomorphism h from $[0, 1)$ to $[½, 1)$ which sends 0 to $½$ and is the identity for t sufficiently close to 1. If we take the Cartesian product of this with the identity on the boundaries, we obtain diffeomorphisms from M and N to M' and N' respectively.

Strictly speaking, the construction of $\mathcal{A}$ depends upon the ***choices of closed collar neighbhorhoods***, but the homeomorphism/diffeomorphism type of $\mathcal{A}$ does not depend upon these choices by the uniqueness of closed collar neighborhoods (since one can always find a homeomorphism/diffeomorphism sending one to the other).

However, the homeomorphism/diffeomorphism type of $\mathcal{A}$ depends very strongly upon the choice of the gluing homeomorphism/diffeomorphism h. To simplify the discussion we shall only consider the topological case. We shall consider two examples.

Example 1. Take M and N to be the cylinder $S^1 \times [-1,1]$, and consider the diffeomorphisms of the boundary $S^1 \times \{-1, 1\}$ sending (z, t) to $(z\varepsilon, -t)$ where

$\varepsilon = \pm 1$. If $\varepsilon = 1$ then one obtains a manifold isomorphic to the torus $S^1 \times S^1$, while if $\varepsilon = -1$ then one obtains the Klein bottle.

Example 2. Take M and N to be the solid torus $S^1 \times D^2$, and consider the two diffeomorphisms of the boundary $S^1 \times S^1$ sending (z, w) to (z, w) and (w, z) respectively. In the first case one obtains a manifold isomorphic to $S^1 \times S^2$ (recall that the sphere is the union of two closed hemispheres which are each isomorphic to D^2), while in the second case one obtains a manifold isomorphic to S^3.

Here is a sketch of the proof: View S^3 as the unit sphere in $\mathbb{C}^2$, and split it into two pieces corresponding to the subsets A, B of complex ordered pairs (z, w) such that $|z| \leq |w|$ and $|z| \geq |w|$ respectively. Then the intersection is the set of all such points such that $|z| = |w| = 1/\sqrt{2}$, and hence it is isomorphic to the torus $S^1 \times S^1$. One then shows that the maps from A and B to $S^1 \times D^2$ sending (z, w) to $|w|^{-1}(w, z)$ in the first case and $|z|^{-1}(z, w)$ in the second are diffeomorphisms.

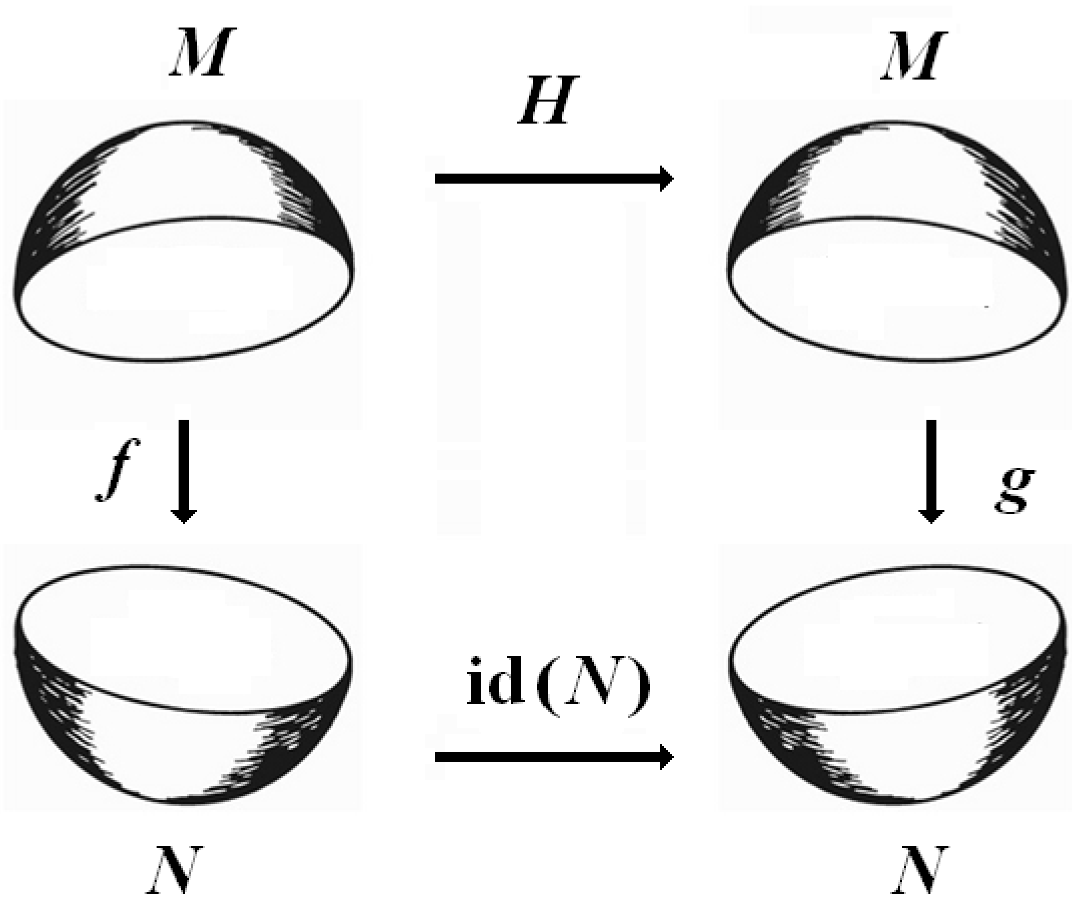
Given two homeomorphic/diffeomorphic bounding manifolds ∂M and ∂N, it is natural to ask whether there are reasonable relations on homeomorphisms or diffeomorphisms $\partial M \to \partial N$ under which the manifolds obtained by gluing along to maps f and g always yield homeomorphic/diffeomorphic manifolds.

Theorem. Let M and N be topological or smooth manifolds with boundaries, and suppose we are given homeomorphisms/diffeomorphisms $f, g : \partial M \to \partial N$. Then the manifolds obtained by gluing M and N along f and g are isomorphic if either of the following hold:

1. The homeomorphism/diffeomorphism $g^{-1}f$ extends to M.
2. The homeomorphisms/diffeomorphisms f and g are isotopic.

In fact, one can weaken the second condition to assuming that f and g are ***concordant*** or ***pseudo – isotopic***; in other words, there is a homeomorphism or diffeomorphism H from $M \times (0, 1)$ to itself such that $H(x, t) = (f(x), t)$ for t close to 0 and $H(x, t) = (g(x), t)$ for t close to 1; if h_t is an isotopy then we may use the associated map $H(x, t) = (h_t(x), t)$ to show that isotopy implies concordance. The canonical extension $H^{\#}$ of the map H to $M \times [0, 1]$ (given by f on $M \times \{0\}$ and g on $M \times \{1\}$) is called a ***concordance*** or a ***pseudo – isotopy***; note that a concordance arises from an isotopy if and only if $p_2 H = p_2$, where p_2 denotes projection onto $[0, 1]$.

Proof of the theorem. We shall derive the second sufficient condition (for concordant isomorphisms) from the first, so suppose that the first condition holds. We can isotop the extension H of $\alpha = g^{-1}f$ so that it is given by a product of the form $\alpha \times \mathbf{identity}$ on the collar neighborhood of the boundary, so let us assume H already satisfies this condition. As suggested by the drawing on the next page, the homeomorphism or diffeomorphism from $M \cup_f N$ to $M \cup_g N$ is defined by H on the interior of M and the identity on the interior of N. To see this map is well – defined, note that the overlap of $\mathrm{INT}(M)$ and $\mathrm{INT}(N)$ is given by $\partial M \times (0, 1) \cong \partial N \times (0, 1)$ such that (x, t) corresponds to $\big(f(x), 1-t\big)$ in the manifold $M \cup_f N$. It follows that the $H(x, t) = \big(\alpha(x), t\big)$ in $\partial M \times (0, 1)$ corresponds to $(x, 1-t)$ in $\partial N \times (0, 1)$ because the point (x, t) corresponds to $\big(g(x), 1-t\big)$ in the manifold $M \cup_g N$,

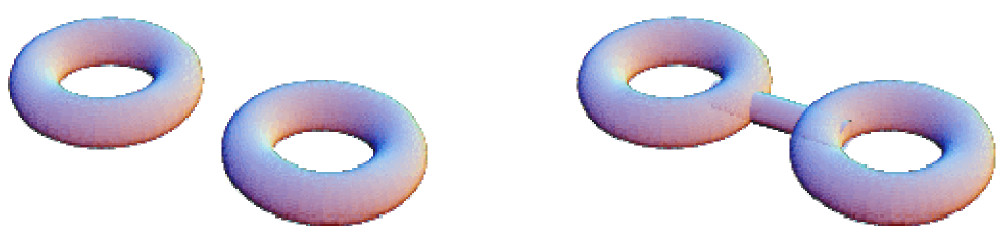


(Modified from **http://www.edupics.com/hemisphere-half-of-a-sphere-t15639.jpg**)

Assume now that f and g are pseudo – isotopic, and let $H^{\#}$ be an extended pseudo – isotopy between them. Then $K^{\#} = (g^{-1} \times \mathbf{identity}_{[0,1]})\, H^{\#}$ defines an extended pseudo – isotopy from $g^{-1}f$ to the identity on ∂M. Using a collar neighborhood of the boundary and the concordance $K^{\#}$ we can construct a diffeomorphism from M to itself which restricts to $g^{-1}f$ on the boundary, is given by the concordance on the collar neighborhood, and is the identity off this collar neighborhood of the boundary.

Connected sums

The idea behind a connected sum of (connected) n – manifolds is simple. We take the disjoint union, remove the interior of a closed disk from each and join the two remaining pieces by a tube.



(Source: http://mathworld.wolfram.com/ConnectedSum.html)

This definition should be viewed skeptically at first, for there are several issues to resolve. The first one is that the right hand manifold does not look like a smooth submanifold where the two ends of the tube meet the complementary pieces. We can overcome this difficulty using the previous gluing construction. Specifically, suppose we are given smooth embeddings $f: D^n \to M$ and $g: D^n \to N$, let M_0 and N_0 denote the complements of the images of $\mathrm{INT}(D^n)$, take h to be the diffeomorphism from ∂M_0 and ∂N_0 corresponding to $g f^{-1}$, and provisionally define a ***connected sum*** $M \,\&\, N$ to be a manifold of the form $M_0 \cup_h N_0$.

The next issue involves the choices of the smooth embeddings f and g. Suppose first that we have new embeddings F and G such that F is isotopic to f and G is isotopic to G. Now let M_1 and N_1 denote the complements of the images of $\mathrm{INT}(D^n)$, take H to be the diffeomorphism from ∂M_1 and ∂N_1 corresponding to GF^{-1}. By the proof of the Cerf – Palais Disk Theorem, there exist extended diffeomorphisms K and L of M and N which are isotopic to the respective identity maps such that $Kf = F$ and $Lg = G$. Using these diffeomorphisms, we can define a new diffeomorphism Ψ from $M_0 \cup_h N_0$ to $M_1 \cup_H N_1$.

If we take arbitrary new embeddings F and G, then the Cerf – Palais Disk Theorem only implies that F is isotopic to either f or $f\,S$, and G is isotopic to either g or $g\,S$, where S is the previously defined hyperplane reflection. In order to complete our analysis of the way M & N depends upon the choices, by the preceding sentence and the previous paragraph it is enough to see what happens if we replace one or both of f or g by $f\,S$ or $g\,S$ respectively. Since S is its own inverse, it follows that if we make ***just one*** replacement then the identification of the boundaries changes from $g\,f^{-1}$ to $g\,S\,f^{-1}$, while if we make ***both*** replacements then the identification of the boundaries is unchanged. Thus we see that the construction is almost uniquely defined, and in all cases ***there are at most <u>two</u> diffeomorphism types that can be realized by the various choices.***

In fact, we shall see that one obtains exactly two diffeomorphism types in some cases; the simplest examples arise when $M = N = \mathbb{CP}^2$. In fact, for this particular case we shall eventually show that ***the two choices for*** $\mathbb{CP}^2$ **&** $\mathbb{CP}^2$ ***are not even homotopy equivalent.***

Orientation considerations

The ambiguity in the preceding construction can be traced back issues involving orientations. We begin with a result which generalizes our previous observations about the Möbius strip.

<u>Proposition.</u> If P is a ***<u>nonorientable</u>*** smooth n – manifold and $f: D^n \to P$ is a smooth embedding, then f is isotopic to $f\,S$, where S is the reflection discussed previously.

<u>Comment on the proof.</u> The key idea in one standard approach is that if P is a nonorientable n – manifold then inside P there is an embedded copy of the open Möbius strip crossed with $\mathbb{R}^{n-2}$. As in the case of the Möbius strip, one can then find a smooth closed n – disk embedding f_0 such that f_0 and $f_0\,S$ are ambient isotopic. In fact, one can assume that the base point of the closed curve in the middle of the Möbius strip is an arbitrary point of P. This proves the proposition for the choice f_0. For any other arbitrary choice f, we know that f is isotopic to either f_0 or $f_0\,S$, which implies that $f\,S$ must be isotopic to either $f_0\,S$ or f_0.

Since we know that f_0 and $f_0 S$ are isotopic, it follows in either case that f and $f S$ must be isotopic.

Corollary. In the notation of the previous subheading, if **either** M or N is nonorientable, then the diffeomorphism type of M & N is independent of all choices.

We have already noted that this result does **not** extend to cases where both manifolds M and N are orientable. Obviously, the next order of business is to analyze this case more carefully. In this discussion we shall use the description of orientations in terms of the canonical double covering $X[+] \to X$ in which the fiber over a point p is given by the two algebraic generators of the homology group $H_n(X,\ X \setminus \{p\}; \mathbb{Z})$, or by the unit $0-$sphere bundle of the $1-$dimensional determinant vector bundle $\Lambda^n \tau_X$. In either case, the orientation is given by a cross section Ω of the double covering $X[+] \to X$ (we have noted that these formulations of the double covering are equivalent). For our purposes, two important properties of the double covering are its naturality with restriction to open subsets, and another is the cylinder identity $X[+] \times \mathbb{R} \ \cong\ (X \times \mathbb{R})[+]$.

One can use the preceding paragraph and the Collar Neighborhood Theorem to show that if (the interior of) a manifold M with boundary is orientable, then the orientation on M determines an orientation of ∂M.

Definition. A homeomorphism (or diffeomorphism) h from a connected oriented manifold (X, Ω_X) to another connected oriented manifold (Y, Ω_Y) is said to be ***orientation – preserving*** provided either of the following equivalent conditions holds:

1. For one (equivalently, for all) points p in X, the induced map h_* from the group $H_n(X, X \setminus \{p\}; \mathbb{Z})$ to $H_n(Y, Y \setminus \{h(p)\}; \mathbb{Z})$ maps the generator $\Omega_X(p)$ in the first (infinite cyclic) group to the generator $\Omega_Y(h(p))$ in the second.
2. If θ_Y is the differential $n-$form on Y such that $<\theta_Y, \Omega_Y>(h(p))$ is positive for all p, then the pullback $h^*\theta_Y$ is a differential form on X such that $<\theta_Y, \Omega_Y>(p)$ is positive for all p.

Before considering the question about connected sums, it will be useful to make a few observations about orientations. The following one contains a converse to the cylinder identity:

Proposition. Suppose that P is a connected smooth manifold such that $P \times \mathbb{R}$ is orientable. Then P is also orientable and there is a $1-1$ correspondence between orientations of P and orientations of $P \times \mathbb{R}$. Furthermore, the reflection map sending (x, t) to $(x, -t)$ defines an orientation – reversing diffeomorphism of $P \times \mathbb{R}$.

Next, we shall illustrate the importance of orientations in cutting and pasting with the following result:

Proposition. In the definition of the gluing construction $\mathcal{a} = M \cup_h N$, suppose that M and N are oriented h is an orientation – **reversing**(!) diffeomorphism from ∂M to ∂N. Then there is an orientation of $\mathcal{a}$ which restricts to the given orientations on the interiors of M_0 and N_0 (we view the latter as open subsets of M and N respectively).

The idea is simple. Suppose we have two oriented manifolds M and N together with an orientation – preserving homeomorphism or diffeomorphism $\varphi : U \to V$ from an open subset of M to an open subset of N. Then we can piece together the orientations of M and N to obtain an orientation of the manifold P formed from gluing M and N together by means of φ. Next, if we are gluing together two bounded manifolds using an orientation – reversing diffeomorphism h of the boundary, then the associated diffeomorphism of collars we note that by a previous result the homeomorphism/diffeomorphism identifying $\partial M \times (0, 1)$ with $\partial N \times (0, 1)$, by sending (x, t) to $\big(h(x), 1-t\big)$, will be the composite of two orientation – reversing homeomorphisms/diffeomorphisms and therefore will be an orientation – preserving homeomorphism/diffeomorphism. It follows that we can piece together the orientations on the interiors of M and N to form an orientation of $\mathcal{a} = M \cup_h N$.

To complete our discussion of connected sums, we need the following version of the Cerf – Palais Disk Theorem for oriented manifolds.

Oriented Cerf – Palais Disk Theorem. Let M be an oriented connected smooth n – manifold without boundary, and let $f, g: D^n \to M$ be smooth orientation – preserving embeddings (it follows that these extend to smooth embeddings on slightly larger open disks; in this context orientation – preserving means that the standard orientation on the thickened disk corresponds to the induced orientation on its image). Then f and g are ambient isotopic.

Note that the extra assumption involving orientations leads to a stronger conclusion with no ambiguity involving reflections.

To derive this from the ordinary Disk Theorem, recall that the latter implies f is ambient isotopic to either g or $g\,S$, where S is the orientation – reversing reflection. Therefore it is enough to check that if f and h are isotopic embeddings into an oriented manifold and f is orientation – preserving, then h is also orientation – preserving. For if we know this, then we know that f cannot be (ambient) isotopic to $g\,S$ because the latter is not orientation – preserving.

We shall prove isotopy invariance using differential forms. If K_t is the isotopy and Ω_M is the orientation structure on M, consider the pullback forms $K_t^{*}\Omega_M$. This is a continuous 1 – parameter family of forms, and it can be written in the form $\lambda(x,t)\theta$, where θ is the standard volume n – form on $\mathbb{R}^n$ and $\lambda(x,t)$ is a smooth function which is never zero. Since $K_0 = f$ is orientation – preserving, it follows that $\lambda(x, 0)$ is positive. Furthermore, if we fix a point x_0, then it also follows by connectedness that the function $\rho(t) = \lambda(x_0, t)$ must also be positive. Once again, using connectedness we conclude that for each t_0 the sign of $\lambda(x, t_0)$ is the same for all x. Combining these, we see that λ is positive everywhere, and therefore for each choice of t the embedding K_t is orientation – preserving.

This finally shows what we want:

1. The diffeomorphism type of the connected sum does not depend upon the choices if at least one of the manifolds M, N is not orientable.
2. The oriented diffeomorphism type will be independent of choices in the oriented case if we stipulate that the disk embeddings must preserve orientations.

These well – defined diffeomorphism types are denoted by $M \;\#\; N$ and called **the** ***connected sum*** of the two manifolds (if at least one of M, N is not orientable) or the oriented connected sum of the two oriented manifolds if both M and N are oriented.

The connected sum constructions have the following important properties:

Theorem. The connected sum constructions are associative and commutative up to diffeomorphism or oriented diffeomorphism (in the oriented case), and the sphere S^n (with the standard orientation in the oriented case) is a two – sided unit for connected sum up to diffeomorphism or oriented diffeomorphism.

A simple argument involving Mayer – Vietoris sequences implies that the homology groups of a connected sum of n – manifolds must satisfy

$$H_k(M \# N) \cong H_k(M) \oplus H_k(N) \quad \text{for} \quad 0 < k < n-1$$

it is clear that most manifolds are not "invertible" with respect to connected sum; specifically, if M has more homology than the sphere in the given dimensions, then it is usually impossible to find some N such that $M \# N$ is diffeomorphic to S^n. As noted in the paper below, if $n \neq 4$ then the class of "invertible" compact manifolds equals the class of smooth manifolds which are almost diffeomorphic to S^n (the so – called ***exotic spheres***). There is some additional discussion of "invertible" compact manifolds in the following paper:

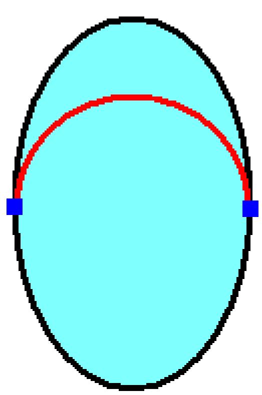
> **J. W. Milnor.** ***Sommes de variétes différentiables et structures différentiables des sphères.*** Bull. Soc. Math. France **87** (1959). 439 – 444.

There is also a more detailed treatment on pages 94 – 94 of the following book:

> **A. Kosinski.** ***Differential manifolds*** (Pure and Applied Mathematics, Vol. 138). Academic Press, Boston MA, 1993.

Modifications for manifolds with boundary. In several contexts it is useful to have a corresponding notion of connected sum for manifolds with boundary. We shall only sketch the construction; further information appears on pages 97 – 99 of the previously cited book by Kosinski.

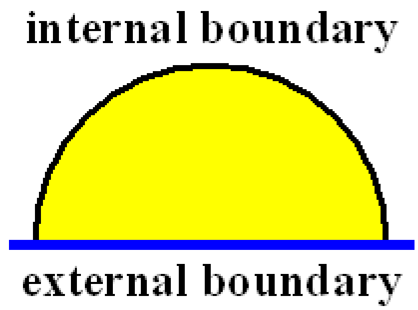
Following standard practice (***e.g.***, see **http://en.wikipedia.org/wiki/Embedding**), an embedding of manifolds with boundary is called a ***proper*** embedding if it sends the boundary of the domain to the boundary of the codomain and the interior of the domain to the interior of the codomain, and in addition the image of the domain is not tangent to the boundary of the codomain. The final condition is included to exclude situations like the embedding of the closed interval $[0, \pi]$ in the solid elliptical region $4x^2 + y^2 \leq 4$ as the semicircular arc $(\cos t, \sin t)$.



The red manifold with boundary is not properly embedded in the solid ellipse. On the other hand, the major and minor axes of the ellipse are properly embedded.

A closely related concept of ***neat submanifold*** is defined and studied in Hirsch (see especially pages 30 – 31).

Define the ***upper half*** $\mathsf{UP}\text{-}D^n$ ***of the closed disk*** D^n to be the set of points whose last coordinates are nonnegative. Topologically $\mathsf{UP}\text{-}D^n$ is a manifold with boundary, and the latter consists of two smooth manifolds with boundary; the ***external boundary*** $\partial_{\mathrm{EXT}}\,\mathsf{UP}\text{-}D^n$ is simply the disk D^{n-1}, while the ***internal boundary*** $\partial_{\mathrm{INT}}\,\mathsf{UP}\text{-}D^n$ is the upper hemisphere D_+^{n-1} of the unit sphere S^{n-1}. The drawing below might be helpful for understanding the notation.



Notice that if M is a manifold with boundary, then every point in the boundary has a closed neighborhood diffeomorphic to $\mathsf{UP}\text{-}D^n$ (in the sense that the map extends to a diffeomorphism over a small neighborhood) such that the given point corresponds to the origin and the intersection of the neighbhorood with ∂M corresponds to $\partial_{\mathrm{EXT}}\,\mathsf{UP}\text{-}D^n$. In the discussion below we shall need a canonical diffeomorphism j from $D_+^{n-1} \times (0, 1)$ to the complement of $\partial_{\mathrm{EXT}}\,\mathsf{UP}\text{-}D^n \cup \{0\}$ in $\mathsf{UP}\text{-}D^n$ which is given by $j(x, t) = tx$ (here juxtaposition means scalar multiplication) .

Notation. We shall say that a smooth embedding of $\mathsf{UP}\text{-}D^n$ in M is ***chartlike*** if it maps $\partial_{\mathrm{EXT}}\,\mathsf{UP}\text{-}D^n$ to ∂M and the complement of the exterior boundary to the interior of M.

One can now state and prove analogs of the Cerf – Palais Theorems for chartlike embeddings if we have a manifold with a connected boundary; the main difference is that the reflection S must be replaced with the diagonal reflection matrix whose diagonal entries are $(-1, 1, \ldots, 1)$. We shall omit the details of proving these extensions of the Cerf – Palais Theorems (as before, there is one version which does not consider orientations and another which does).

We can now define a ***boundary connected sum*** of bounded manifolds $(M, \partial M)$ and $(N, \partial N)$, provided the manifolds are connected and have connected boundaries.

Specifically, take chartlike embeddings f: UP-$D^n \to M$ and g: UP-$D^n \to N$ (which preserve orienetations in the oriented case) and construct the quotient of the disjoint union of $M - f[\{0\}]$ and $N - g[\{0\}]$ modulo identification of $jf(x, t)$ with $jg(x, 1-t)$. As before, the diffeomorphism (or oriented diffeomorphism) type of this manifold does not depend upon the choices, and it is called the boundary connected sum. Generally these manifolds are denoted by notation like $(M, \partial M)$ # $(N, \partial N)$, but in some papers and books they are denoted by the symbol M ♮ N (*musical natural sign*).

As in the unbounded case, this connected sum is associative and commutative up to diffeomorphism (orientation – preserving in the oriented case), and the n – disk D^n turns out to be a two – sided identity up to diffeomorphism. Furthermore, by construction the boundary of $(M, \partial M)$ # $(N, \partial N)$ is canonically diffeomorphic (with appropriate orientations in the oriented case) to ∂M # ∂N.

Of course, one can make similar constructions if the boundaries of the manifolds are not connected, but then one must specify which components are being glued together. None of this is difficult, but it does require time and effort to write things out precisely.