Survey of differential forms

This is a summary of the main points of the theory. Additional details can be found in Conlon or on pages 245–288 of Little Rudin.

1 : The basic definitions

Differential forms provide a convenient and powerful setting for generalizing classical vector analysis to higher dimensions, and they have numerous uses in both mathematics and physics. Setting up the theory requires some time and effort, but differential forms can be used very effectively to unify and simplify some fundamentally important concepts and results. They have become the standard framework for analyzing an extremely wide range of topics and problems.

Covariant tensors and differential forms

Let U be an open subset of $\mathbf{R}^n$, and let p be a nonnegative integer. A **covariant tensor field of rank** p is defined to be an expression of the form

$$\sum_{i_1, i_2,\ (etc.)} g_{i_1\, i_2\, \cdots\, i_p}\, dx^{i_1} \otimes \ \cdots \ \otimes dx^{i_p}$$

where

(1) each $g_{i_1\, i_2\, \cdots\, i_p}$ is a smooth real valued function on U,

(2) each i_j ranges from 1 to n,

(3) two expressions are equal if and only if the functional coefficients of each $dx^{i_1} \otimes \cdots \otimes dx^{i_p}$ are equal.

We shall call denote this object by $\mathbf{Cov}_p(U)$. It will be understood that $\mathbf{Cov}_0(U) = \mathcal{C}^\infty(U)$; note also that there is a natural identification of $\mathbf{Cov}^1(U)$ with the space of differential 1-forms we considered in Section V.3 of the lecture notes.

The space of **exterior** or **differential** p**-forms** on U is defined to be the quotient of $\mathbf{Cov}_p(U)$ obtained by the identification

$$dx^{i_1} \otimes \ \cdots \ \otimes dx^{i_p} \quad \approx \quad -\, dx^{j_1} \otimes \ \cdots \ \otimes dx^{j_p}$$

if $[\, j_1\, i_2\ \cdots\ j_p\,]$ is obtained from $[\, i_1\, i_2\ \cdots\ i_p\,]$ by switching exactly two of the terms, say i_s and i_t where $s \neq t$. If $i_s = i_t$ for some $s \neq t$ then this is understood to imply that $dx^{i_1} \otimes \ \cdots \ \otimes dx^{i_p}$ is equal to its own negative, and since we are working with real vector spaces this means that the expression in question is identified with zero. The set of all differential p-forms on an open subset $U \subset \mathbf{R}^n$ is denoted by $\wedge^p(U)$, and the images of the basic objects in if $dx^{i_1} \otimes \ \cdots \ \otimes dx^{i_p}$ is one of the basic objects in $\mathbf{Cov}_p(U)$ as above, then its image in $\wedge^p(U)$ is denoted by

$$dx^{i_1} \wedge \ \cdots \ \wedge dx^{i_p} \ .$$

By convention we also set $\wedge^0(U)$ equal to $\mathcal{C}^\infty(U)$.

PROPOSITION. *If $p > n$ then $\wedge^p(U) = 0$, and if $0 < p \leq n$ then every element of $\wedge^p(U)$ can be written uniquely as a linear combination of the basic forms*

$$dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

with coefficients in $\mathcal{C}^\infty(U)$, where the indexing sequences $\{ i_j \}$ satisfy $i_1 < \cdots < i_p$.

This is an immediate consequence of the construction.■

If $p = 1$ then the definition of $\wedge^1(U)$ is equivalent to the previous one involving sections of the cotangent bundle.

Integrals defined by differential forms

The motivation for the definition comes from the use of differential 1-forms as the integrands of line integrals. In particular, we would like 2-forms to represent the integrands of surface integrals and n-forms to represent the integrands of ordinary (Riemann or Lebesgue) integrals over appropriate subsets of U. Note in particular that if U is open in $\mathbf{R}^n$, then every element of $\wedge^n(U)$ is uniquely expressible as

$$h(x) \cdot dx^1 \wedge \cdots \wedge dx^n$$

for some $h \in \mathcal{C}^\infty(U)$.

So how do we form integrals such that the integrand is a p-form and the construction reduces to the usual ones for line and surface integrals if $p = 1$ or 2? The key is to notice that such integrals are first defined using parametric equations for a curve or surface defined for all values of the variable(s) in some open subset of $\mathbf{R}$ or $\mathbf{R}^2$.

Following Rudin, we do so by defining a *smooth singular p-surface piece* in U to be a continuous map $\sigma : \Delta \to U$ such that Δ is compact in $\mathbf{R}^p$ and σ extends to a smooth function on an open neighborhood of Δ in $\mathbf{R}^p$. In multivariable calculus one genreally assumes also that the extension of σ to an open set is a *smooth immersion*, or at least this is true if one subdivides the domain of definition into suitable pieces and permits bad behavior at boundary points of such pieces, but we shall not make any such assumptions on the rank of $D\sigma$ in these notes.

For each object σ as in the previous paragraph and each tensor $\Lambda \in \mathbf{Cov}_p(U)$ we can define an integral by the following formula:

$$\int_\sigma \Lambda \quad = \quad \int_\sigma \sum_{i_1, i_2, \ etc.} g_{i_1\, i_2\, \cdots\, i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p} \quad =$$

$$\sum_{i_1, i_2, \ etc.} \int_\Delta g_{i_1\, i_2\, \cdots\, i_p} \circ \sigma(u) \frac{\partial(x^{i_1}, \ \cdots \ , x^{i_p})}{\partial(u^1, \ \cdots \ , u^p)}$$

As usual, expressions of the form

$$\frac{\partial(x_a, \ \cdots \ )}{\partial(u_1, \ \cdots \ )}$$

represent Jacobian determinants. We then have the following key observation which allows us to work with forms rather than tensors:

PROPOSITION. *In the integral above, the value only depends upon the image λ of Λ in $\wedge^p(U)$.*

Proof. It suffices to consider simple integrands consisting of only one summand. For each sequence

$$x^{i_1}, \cdots, x^{i_p}$$

we need to show that if we switch two terms x^a and x^b then the sign of the integral changes if dx^a and dx^b are both factors of the integrand. The effect of making such a change on the integrand is to switch two columns in the $p \times p$ matrix of functions whose determinant is the Jacobian

$$\frac{\partial(x^{i_1}, \cdots, x^{i_p})}{\partial(u^1, \cdots, u^p)}$$

and we know this operation changes signs; this proves the point that we need to reach the conclusion of the proposition.■

Because of the preceding result WE SHALL ASSUME HENCEFORTH THAT INTEGRANDS ARE DIFFERENTIAL p-FORMS.

2 : Operations on differential forms

There are several fundamental constructions on differential forms that are used extensively.

Exterior products

It follows immediately from the definitions that each $\wedge^p(U)$ is a real vector space and in fact is a module over $\mathcal{C}^\infty(U)$ However, there is also an important multiplicative structure that we shall now describe. We shall begin by defining a version of this structure for covariant tensors. Specifically, there are $\mathcal{C}^\infty(U)$-bilinear maps

$$\otimes : \mathbf{Cov}_p(U) \times \mathbf{Cov}_q(U) \longrightarrow \mathbf{Cov}_{p+q}(U)$$

sending a pair of monomials

$$\left(g_{i_1\, i_2\, \cdots\, i_p}\, dx^{i_1} \otimes \cdots \otimes dx^{i_p} \; , \; h_{j_1\, j_2\, \cdots\, j_q}\, dx^{j_1} \otimes \cdots \otimes dx^{j_q} \right)$$

to the monomial

$$g_{i_1\, i_2\, \cdots\, i_p} h_{j_1\, j_2\, \cdots\, j_q} \cdot dx^{i_1} \otimes \cdots \otimes dx^{i_p} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q} \; .$$

In order to show this passes to a $\mathcal{C}^\infty(U)$-bilinear map

$$\wedge_{p,q} : \wedge^p(U) \times \wedge^q(U) \longrightarrow \wedge^{p+q}(U)$$

we need to show that if $\xi \in \mathbf{Cov}_p(U)$ and $\eta \in \mathbf{Cov}_q(U)$ are monomials as above and ξ' and η' are related to ξ and η as in the definition of differential forms, then the images of $\otimes(\xi, \eta)$ and $\otimes(\xi', \eta')$ are equal. As above we are assuming

$$\xi \quad = \quad g_{i_1\, i_2\, \cdots\, i_p}\, dx^{i_1} \otimes \cdots \otimes dx^{i_p} \quad , \quad \eta \quad = \quad h_{j_1\, j_2\, \cdots\, j_q}\, dx^{j_1} \otimes \cdots \otimes dx^{j_q} \; .$$

Since two covariant monomial tensors determine the same differential form if they are related by a finite sequence of elementary moves (permuting the dx^q's or replacement by zero if there is a repeated such factor), it is enough to show that one obtains the same differential form provided ξ' and η' are related to ξ and η by a single elementary move (which affects one form but not the other).

Suppose the elementary move switches two variables; then we may write

$$\xi' \quad = \quad \alpha \cdot g_{k_1\, k_2\, \cdots\, k_p}\, dx^{k_1} \otimes \ \cdots \ \otimes dx^{k_p} \qquad , \qquad \eta' \quad = \quad \beta \cdot h_{\ell_1\, \ell_2\, \cdots\, \ell_q}\, dx^{\ell_1} \otimes \ \cdots \ \otimes dx^{\ell_q}$$

where $\{k_1\, k_2\ \cdots\ k_p\}$ and $\{\ell_1\, \ell_2\ \cdots\ \ell_q\}$ are obtained from $\{i_1\, i_2\ \cdots\ i_p\}$ and $\{j_1\, j_2\ \cdots\ j_q\}$ either by doing nothing or by switching two of the variables and the coefficients α and β are ± 1 depending upon whether or not variables were switched in each case. From this description one can check directly (with some tedious computations) that the images of $\otimes(\xi, \eta)$ and $\otimes(\xi', \eta')$ in $\wedge^{p+q}(U)$ are equal. On the other hand, if one has repeated factors in either ξ or η and the corresponding object ξ' or η' is zero, then it is immediately clear that $\otimes(\xi, \eta)$ and $\otimes(\xi', \eta')$ in $\wedge^{p+q}(U)$ both zero and hence are equal.

PROPOSITION. *If $\theta \in \wedge^p(U)$ and $\omega \in \wedge^q(U)$, then we have $\theta \wedge \omega = (-1)^{pq}\, \omega \wedge \theta$.*

Proof. Using bilinearity we may immediately reduce this to the special case where

$$\theta \quad = \quad dx^{i_1} \wedge \ \cdots \ \wedge dx^{i_p} \qquad , \qquad \omega \quad = \quad dx^{j_1} \wedge \ \cdots \ \wedge dx^{j_q} \ .$$

In this case we have

$$\theta \wedge \omega \quad = \quad dx^{i_1} \wedge \ \cdots \ \wedge dx^{i_p} \wedge dx^{j_1} \wedge \ \cdots \ \wedge dx^{j_q} \quad , \quad \omega \wedge \theta \quad = \quad dx^{j_1} \wedge \ \cdots \ \wedge dx^{j_q} \wedge dx^{i_1} \wedge \ \cdots \ \wedge dx^{i_p} \ .$$

Therefore we need to investigate what happens if one rearranges the variables using some permutation.

If γ is an arbitrary permutation then γ is a product of transpositions, and therefore it follows that if one permutes variables by γ the effect on a basic monomial form is multiplication by $\mathrm{sgn}(\gamma)$. Therefore the proof of the formula in the proposition reduces to computing the sign of the permutation which takes the first p numbers in $\{1, \ \cdot\ , p+q\}$ to the last p numbers in order and takes the last q numbers to the first q numbers in order. It is an elementary combinatorial exercise to verify that the sign of this permutation is pq (*e.g.*, fix one of p or q and proceed by induction on the other).■

The following property is also straightforward to verify, and in fact it is a consequence of the analogous property for covariant tensors:

PROPOSITION. *If θ and ω are as above and $\lambda \in \wedge^r(U)$, then one has the associativity property $(\theta \wedge \omega) \wedge \lambda = \theta \wedge (\omega \wedge \lambda)$.*■

Exterior derivatives

We have already seen that there is a well-defined map $d : \wedge^0(U) \to \wedge^1(U)$ defined by taking exterior derivatives, and in fact for each p one can define an exterior derivative

$$d^p : \wedge^p(U) \ \longrightarrow \ \wedge^{p+1}(U) \ .$$

These maps are linear transformations of real vector spaces and are defined on monomials by the formula

$$d\big(\ g\,dx^{i_1}\wedge\ \cdots\ \wedge dx^{i_p}\ \big)\quad=\quad dg\wedge\ dx^{i_1}\wedge\ \cdots\ \wedge dx^{i_p}\ .$$

If we take $g=1$ the preceding definition implies

$$d\big(\ dx^{i_1}\wedge\ \cdots\ \wedge dx^{i_p}\ \big)\quad=\quad 0\ .$$

One then has the following basic consequences of the definitions.

THEOREM. *The exterior derivative satisfies the following identities:*

(*i*) *If θ is a p-form then $d(\theta\wedge\omega)\ =\ (d\theta)\wedge\omega\ +\ (-1)^p\theta\wedge(d\omega)$.*

(*ii*) *For all λ we have $d(d\lambda)=0$.*

Sketch of proof. In each case one can use linearity or bilinearity to reduce everything to the special case of forms that are monomials. For examples of this type it is a routine computational exercise to verify the identities described above.■

Definition. A differential form ω is said to be *closed* if $d\omega=0$ and exact if $\omega=d\lambda$ for some λ. The second part of the theorem implies that exact forms are closed. On the other hand, the 1-form

$$\frac{y\,dx-x\,dy}{x^2+y^2}$$

on $\mathbf{R}^2-\{0\}$ is closed but not exact.

Change of variables (pullbacks)

The pullback construction on 1-forms extends naturally to forms of higher degree. Specifically, if V is open in $\mathbf{R}^m$ and $f:V\to U$ is smooth then there are real vector space homomorphisms $f^*:\wedge^p(U)\to\wedge^p(U)$ that are defined on monomials by the formula

$$f^*\big(\ g\,dx^{i_1}\wedge\ \cdots\ \wedge dx^{i_p}\ \big)\quad=\quad(g\circ f)\,df^{i_1}\wedge\ \cdots\ \wedge df^{i_p}$$

where f^i denotes the $i^{\rm th}$ coordinate function of f. If $p=1$ this coincides with the previous definition.

The next result implies that the pullback construction preserves all the basic structure on exterior forms that we defined above and it has good naturality properties:

THEOREM. (*i*) *In the notation above we have $f^*(\theta\wedge\omega)=f^*\theta\wedge f^*\omega$ and $f^*\circ d\lambda=d\circ f^*\lambda$.*

(*ii*) *The pullback map for id_U is the identity on $\wedge^p(U)$, and if $h:W\to V$ is another smooth map, then $(f\circ h)^*=h^*\circ f^*$.*

(*iii*) *The pullback maps and exterior derivatives satisfy the compatibility relations $d\circ f^*=f^*\circ d$.*

Complete derivations of these results appear on pages 263–264 of Rudin.■

The pullback also has the following basic compatibility property with respect to integrals:

CHANGE OF VARIABLES FOR INTEGRALS. *Let $\omega\in\wedge^p(U)$, let $f:V\to U$ be smooth, and let $\sigma:\Delta\to V$ be a smooth p-surface. Then integration of differential forms satisfies the following change of variables formula:*

$$\int_\Delta f^*\omega\quad=\quad\int_{f\,\circ\,\sigma}\omega$$

A derivation of this formula appears on pages 264–266 of Rudin.■

3 : Globalization to smooth manifolds

We shall now indicate how the entire theory of differential forms extends to smooth manifolds. (This draws more heavily on other material from Mathematics 205C).

Exterior powers of a vector bundle

As in many other cases, the vector bundle construction comes from a construction on vector spaces which we shall now outline. If V is a vector space over the field $\mathbf{k}$, then one has exterior power vector spaces $\wedge^p(V)$ that are spanned by objects of the form $v_1 \wedge \cdots \wedge v_p$ where $v_j \in V$. As in the case of exterior forms we set one such monomial equal to the negative of the other if the factors of one are obtained from the factors of another by a single transposition, and if $1+1=0$ in $\mathbf{k}$ we also set a monomial equal to zero if it has two identical factors (this follows from the first condition in the other cases). The wedge construction is multilinear, and there are wedge products similar to those for differential forms. Some work is needed to justify all these assertions; details will appear in the full version of Section V.4 of the notes. Likewise one has the relationship

$$\dim V \;=\; n \quad\Longrightarrow\quad \dim \wedge^p(V) \;=\; \binom{n}{p}$$

and if $\{\, x_i \,\}$ is a basis for V then $x_{1_1} \wedge \cdots \wedge x_{i_p}$ is a basis for $\wedge^p(V)$. Given a linear transformation $T : V \to W$ one has associated linear transformations $\wedge^p(T)$ that have the standard naturality properties:

(1) $\wedge^p(I) \;=\; I$

(2) $\wedge^p(S \circ T) \;=\; \wedge^p(S) \circ \wedge^p(T)$

(3) If T is invertible then so is $\wedge^p(T)$ and $\wedge^p(T)^{-1} \;= \wedge^p(T^{-1})$.

Taken together these define a homomorphism

$$\wedge^p : GL(n, \mathbf{k}) \longrightarrow GL\left(\binom{n}{r}, \mathbf{k}\right)$$

such that for each invertible matrix A the entries of $\wedge^p(A)$ are polynomials in the entries of A. If $\mathbf{k}$ is the real or complex numbers this is enough to construct associated exterior power bundles $\wedge^p(\xi)$ associated to a continuous or smooth vector bundle ξ.

Definition. If M is a smooth manifold, then a **differential p-form** on M is a smooth cross section of $\wedge^p(\tau_M^*)$.

Locally these correspond to our previous definitions of differential forms.

Constructing differential forms from local data

We begin by reformulating our earlier result on this question for differential 1-forms.

LEMMA. *Let $f : U \to V$ be a diffeomorphism between open subsets of $\mathbf{R}^n$, let $f_* : \wedge^1(U) \to \wedge^1(V)$ be the direct image map, and $f^* : \wedge^1(V) \to \wedge^1(U)$ be the pullback map. Then f^* and f_* are inverse to each other.*

This is an immediate consequence of the definitions.■

We can now give a criterion for fitting together differential forms that are defined locally.

CONSTRUCTION OF DIFFERENTIAL FORMS. *Let M be a smooth manifold with an atlas of smooth charts (U_α, h_α) whose transition maps are given by $\psi_{\beta\alpha} : V_{\beta\alpha} \to V_{|alpha\beta}$. Suppose that we are given forms $\omega_\alpha \in \wedge^p(U_\alpha)$ such that*

$$\psi^*_{\beta\alpha}\omega_\beta | V_{\alpha\beta} \quad = \quad \omega_\alpha | V_{\beta\alpha}$$

for all α and β. Then there is a unique p-form $\omega \in \wedge^p(M)$ such that $h^*_\alpha \omega = \omega_\alpha$ for all α.■

Thanks to this result and the properties of the pullback maps we can generalize all the algebraic and differential structure that had been defined for p-forms over open subsets of Euclidean spaces, and we can generalize all the identities that were established for forms over these open sets.

4 : Relation to classical vector analysis

We shall now explain how the basic constructions and main theorems of vector analysis can be expressed in terms of differential forms. For most of this section U will denote an open subset of $\mathbf{R}^3$.

Let $\mathbf{X}(U)$ be the Lie algebra of smooth vector fields on U. As a module over $\mathcal{C}^\infty(U)$ the space of vector fields is isomorphic to each of $\wedge^1(U)$ and $\wedge^2(U)$, and $\mathcal{C}^\infty(U)$ is isomorphic to $\wedge^3(U)$; recall that $\mathcal{C}^\infty(U) = \wedge^0(U)$ by definition. For our purposes it is important to give specific isomorphisms $\Phi^1 : \mathbf{X}(U) \to \wedge^1(U)$, $\Phi^2 : \mathbf{X}(U) \to \wedge^2(U)$, $\Phi^3 : \mathcal{C}^\infty(U) \to \wedge^3(U)$. A vector field will be viewed as a vector valued function $\mathbf{V} = (F, G, H)$ where each of F, G, H is a smooth real valued function on U.

$$\Phi_1(F, G, H) \quad = \quad F\,dx + G\,dy + H\,dz$$

$$\Phi_2(F, G, H) \quad = \quad F\,dy \wedge dx + \; G\,dz \wedge dx + H\,dx \wedge dy$$

$$\Phi_3(f) \quad = \quad f\,dx \wedge dy \wedge dx$$

We then have the following basic relationships:

$$(i) \quad \nabla f \quad = \quad \Phi_1^{-1}(df)$$

$$(ii) \quad \mathbf{curl}(\mathbf{V}) \quad = \quad \Phi_2^{-1} \circ d \circ \Phi_1(\mathbf{V})$$

$$(iii) \quad \mathbf{div}(\mathbf{V}) \quad = \quad \Phi_3^{-1} \circ d \circ \Phi_2(\mathbf{V})$$

Each of these is a routine computation.

From this perspective the vector analysis identities

$$\mathbf{curl}(\nabla f) = 0 \quad , \quad \mathbf{div}\,\mathbf{curl}(\mathbf{V}) \quad = \quad 0$$

are equivalent to special cases of the more general relationship $d \circ d = 0$.

More important, the identifications in $(i)-(iii)$ lead to a general statement that includes the following three basic results:

(1) The standard path independence result stating that the line integral $\int \nabla f \cdot d\mathbf{x}$ is equal to $f(\text{final point on curve}) - f(\text{initial point on curve})$.

(2) Stokes' Theorem (*note the spelling!!*) relating line and surface integrals.

(3) The so-called Gauss or Divergence Theorem relating surface and volume integrals.

In each case the result can be stated in terms of differential forms and p-surfaces (where $p = 1, 2, 3$) as follows: *If we are given a p-surface σ that has a reasonable notion of boundary $\partial\sigma$ such that $\partial\sigma$ is somehow a sum of $(p-1)$-surfaces with coefficients of ± 1, then*

$$\int_{\partial\sigma} \omega \quad = \quad \int_{\sigma} d\omega$$

for all $(p-1)$-forms ω.

These are also special cases of a more general result.

GENERALIZED STOKES' FORMULA. *The formula above is valid for open subsets in $\mathbf{R}^n$ and for arbitrary values of n where p is an arbitrary positive number $\leq n$.*

Clearly it is eventually important to be specific about what sorts of pieces are reasonable in the description of boundary. Certainly any such concept should include the intuitive notions of boundary for the disk $D^p \subset \mathbf{R}^p$ and the hypercube $[-1,1]^p \subset \mathbf{R}^p$. It turns out that one wants objects whose fundamental building blocks are suitably positioned diffeomorphic images of the simplex

$$(0 * \Delta)_p \quad \subset \quad \mathbf{R}^p$$

consisting of all $(x_1, ! \cdots, x_p) \in \mathbf{R}^p$ such that $x_i \geq 0$ for all i and $\sum_i x_i \leq 1$; this object is called Q_p on page 247 of Rudin. Unfortunately, an attempt to describe terms like "building blocks" and "suitably positioned" would require some lengthy digressions into topology that are beyond the scope of Mathematics 205C, so we are going to stop here. **However, these issues will be studied further in Mathematics 246B.**

A derivation of Stokes' formula for the building blocks described above in Rudin on pages 272–275.

For more general cases, one uses a decomposition of the p-surface into such building blocks to obtain a similar conclusion. In spirit this is similar to the procedure one uses to generalize a result like Green's Theorem in the plane. Proving the result for the standard right triangle whose vertices are the origin and the standard unit vectors is straightforward, and one splits a general region with boundary into finitely many pieces that are diffeomorphic images of this triangle to recover the general result.■