Comparing smooth and homological orientations

In courses on smooth manifolds, one usually sees orientations defined in terms of tangent bundles, smooth atlases or differential forms of maximum dimension. On the other hand, algebraic topology books generally describe orienetability in terms of certain phenomena involving relative homology groups. Our purpose here is to link these two approaches for smooth manifolds. The main idea can be stated very simply: Given a diffeomorphism h between open subsets of $\mathbb{R}^n$, there is an entirely homological characterization of the sign of the Jacobian of h.

We shall use the following references as background for our discussion:

> **J. W. Vick.** *Homology Theory: An Introdution to Algebraic Topology* (Springer Graduate Texts in Mathematics Vol. 145), *Springer-Verlag, New York* [*etc.*], 1994. ISBN: 0–387–94126–6.
>
> **H. Samelson.** *On Poincaré duality*, Journal d'Analyse Mathématique **14** (1965), 323–336.
>
> **A. Hatcher.** *Algebraic Topology* (Third Paperback Printing), *Cambridge University Press, New York NY*, 2002. ISBN: 0–521–79540–0.

One important fact about Hatcher's book is that it can be legally downloaded from the Internet at no cost for personal use; and here is the link to the online version:

`www.math.cornell.edu/∼hatcher/AT/ATpage.html`

There is a far more classical approach to orientatability and duality theorems in the following notes:

`http:math.ucr.edu/∼res/math246B/simplicial-duality.pdf`

In particular, the cited notes provide detailed information on the motivational discussions involving *dual cells* on pages 143–146 of Vick and pages 232-233 of Hatcher.

Orientations of certain homology groups

The first step is to construct a canonical set of generators $\omega[S^n]$ for the integral homology groups $H_n(S^n;\mathbb{Z})$.

THE CASES $n = 0, 1$. Take $\omega[S^0]$ to be the homology class of the singular 0-chain given by $\langle -1\rangle - \langle +1\rangle$, and take $\omega[S^1]$ to be the homology class of the singular 1-simplex $T : \Delta_1 \to S^1$ such that

$$T\big(\, u\mathbf{e}_0 + (1-u)\mathbf{e}_1 \,\big) \quad = \quad (\cos 2\pi u, \sin 2\pi u)\ .$$

OBTAINING $\omega[S^{n+1}]$ FROM $\omega[S^n]$ RECURSIVELY WHEN $n \geq 1$. As usual we write S^{n+1} as a union of the upper and lower hemispheres D^{n+1}_+ and D^{n+1}_-. We then have the following chain of isomorphisms:

$$H_n(S^n) \ \leftarrow\ H_{n+1}(D^{n+1}_+, S^n) \ \rightarrow\ H_{n+1}(S^{n+1}, D^{n+1}_-) \ \leftarrow\ H_{n+1}(S^{n+1})$$

The left hand isomorphism comes from the exact homology sequence of the pair (D^{n+1}_+, S^n), the center isomorphism is the usual excision-like isomorphism induced by inclusion of pairs, and the right hand isomorphism comes from the exact homology sequence of the pair (S^{n+1}, D^{n+1}_-).

The preceding immediately yields preferred generators of the groups $H_n(\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{0}\})$ using the canonical isomorphism of the latter with the reduced homology $\widetilde{H_n}(S^{n-1})$. We shall call these classes $[\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{0}\}]$.

One reason for considering such classes is the following far-reaching consequence excision and homotopy invariance:

PROPOSITION. *If M is a topological n-manifold and $p \in M$, then $H_n(M, M - \{p\})$ is isomorphic to $H_n(\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{0}\})$ for all choices of coefficients.*

Proofs of this result are given in Vick and Hatcher.■

Local degrees of homeomorphisms

Throughout the discussion below, the diagonal of a Cartesian product of the form $X^2 = X \times X$ will be denoted by X (we suppress the subscript to simplify the notation).

The following result is the starting point:

PROPOSITION. *Let $\mathbf{E} = \mathbb{R}^n$. Then the inclusion*

$$(\mathbf{E}, \mathbf{E} - \{\mathbf{0}\}) \quad \cong \quad (\mathbf{E} \times \{\mathbf{0}\}), \mathbf{E} - \{\mathbf{0}\}) \times \{\mathbf{0}\}) \quad \rightarrow \quad (\mathbf{E}^2, \mathbf{E}^2 - \Delta)$$

induces isomorphisms in homology.

This follows from the fact that the shear map of $\mathbf{E}^2$ sending (x, y) to $(x - y, y)$ induces a homeomorphism of pairs from $(\mathbf{E}^2, \mathbf{E}^2 - \Delta)$ to $(\mathbf{E}^2, (\mathbf{E} - \{\mathbf{0}\}) \times \mathbf{E})$.■

Using this result we shall define the *standard global orientation* of $\mathbf{E} = \mathbb{R}^n$ to be the generator $\mathbf{U}$ of $H_n(\mathbf{E}^2, \mathbf{E}^2 - \Delta)$ which is the image of the standard generator $[\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{0}\}]$. The class $\mathbf{U}$ in turn defines preferred generators for each of the groups $H_n(\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{x}\})$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Specifically, we can choose $[\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{x}\}]$ such that this class maps to $\mathbf{U}$ under the slice inclusion from $\mathbf{x} \times (\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{x}\})$ to $(\mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n \times \mathbb{R}^n - \Delta)$.

By excision, the if V is open in $\mathbb{R}^n$ and $\mathbf{x} \in V$, we have an isomorphism from $H_*(V^2, V^2 - \{\mathbf{x}\})$ to $H_n(\mathbb{R}^n, \mathbb{R}^n - \{\mathbf{x}\})$, and using this isomorphism we can define standard a standard generator $\mathbf{U}_{x,V}$ of $H_n(V^2, V^2 - \{\mathbf{x}\})$.

Definition. Suppose that V and W are open subsets of $\mathbb{R}^n$ and $h : V \to W$ is continuous and 1–1, and let $x \in V$. Then the *local degree* of h at x is the integer ε such that h maps the generator $\mathbf{U}_{x,V}$ to $\varepsilon \, \mathbf{U}_{h(x),W}$. Note that the 1–1 condition implies that h defines a continuous map from $(V, V - \{x\})$ to map from $(W, W - \{h(x)\})$. Furthermore, note that if h is a homeomorphism then $\varepsilon = \pm 1$.

Local degrees have the following basic properties:

(1) The local degree is constant if the domain is connected.

(2) The local degree is $+1$ if the map in question is an inclusion. [*Note:* If we combine this with Invariance of Domain, it follows that the local degree will always be ± 1.]

(3) The local degree is $+1$ if the map in question is a translation.

(4) The local degree is the sign of the determinant if the map in question is an invertible linear transformation and $\mathbf{0}$ is the reference point.

(5) The local degree is the sign of the Jacobian if the map in question is a smooth map whose derivative is invertible everywhere. [By the preceding properties it is enough to do this for maps which send $\mathbf{0}$ to itself and whose derivatives there are the identity; in such cases one can use an argument involving the multivariable Taylor's Formula to show that the given map is homotopic to the identity by a homotopy such that the tracks of nonzero points miss the origin.]

We then have the following basic result:

THEOREM. *A topological manifold M is orientable in the sense of algebraic topology if an only if it has an atlas such that the local degrees of the transition homeomorphisms are always equal to* $+1$.

This result and observation (5) link the concept of orientation in topology with the usual concept(s) of orientation for smooth manifolds as in Theorem 25 on page 28 of the following online reference:

`http://www.math.ucla.edu/~petersen/manifolds.pdf`