LECTURES ON SEIFERT MANIFOLDS

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AND

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Contents

Preface

This volume comprises notes of a course given by the second author at Brandeis University in Spring, 1981. The notes were written by the first author. The course ended with a discussion of geometric structures on Seifert manifolds and relations to quasihomogeneous complex surface singularities. This material here appears as an appendix which is a reprint, by permission of the American Mathematical Society, of an article in Proceedings of Symposia in Pure Mathematics, volume 40.

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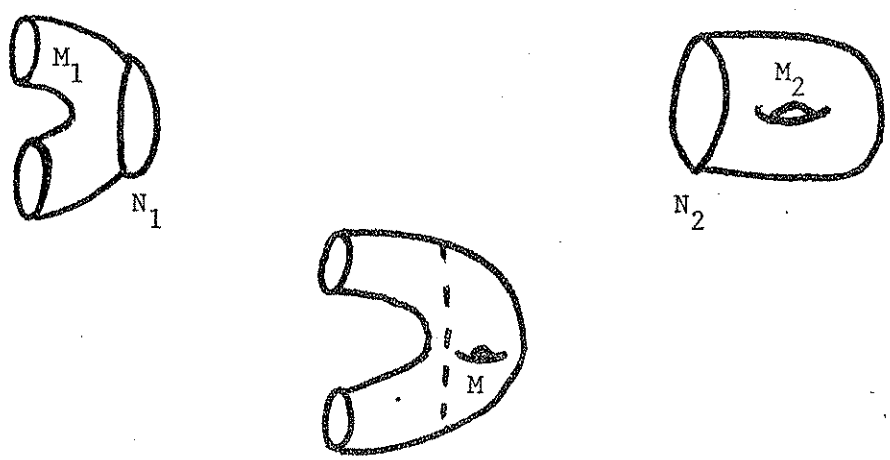
Introduction

In this chapter we give, mostly without proof, some necessary topological preliminaries. Proofs of these theorems can be found in most books on differential topology (e.g. M. Hirsch: Differential Topology).

The basic construction used is cutting and pasting of manifolds. Given two smooth manifolds M_1 and M_2 with $N_i \subseteq \partial M_i$ as a component $(i = 1,2)$ and $h: N_1 \to N_2$ a diffeomorphism, we can form $M = M_1 \cup_h M_2 = (M_1 + M_2)/(x) \equiv h(x))$ (see figure) where $+$ denotes disjoint union.



The following theorem shows this construction can be done in the smooth category.

Theorem 0.1.

a) M can be given a smooth structure such that M_1 and M_2 are smooth submanifolds.

b) Given two such smooth structures on M, say $\mathscr{A}_1$ and $\mathscr{A}_2$, there exists a diffeomorphism $h: (M,\mathscr{A}_1) \to (M,\mathscr{A}_2)$ (where $M,\mathscr{A}_i)$ is M with smooth structure $\mathscr{A}_i, i = 1,2)$ such that

i) h is arbitrarily close to the id.

ii) $h = id$ outside an arbitrarily small neighborhood of N_1

iii) h is isotopic to the id through homeomorphisms satisfying i) and ii)

iv) In i) - iii) one can assume $h|M_1 = id$ or $h|M_2 = id$.

c) If $h' = f_2 \circ h \circ f_1$ where $f_i: N_i \to N_i$ $(i = 1,2)$ is a diffeomorphism which extends to a diffeomorphism: $M_i \to M_i$ $(i = 1,2)$. Then $M_1 \cup_h M_2 \underset{C^\infty}{\cong} M_1 \cup_{h'} M_2$.

<u>Proposition 0.2.</u> Let $M_1, M_2, N_1, N_2, h: N_1 \to N_2$ be as in the previous theorem. If $h': N_1 \to N_2$ is isotopic to h then $M_1 \cup_h M_2 \underset{C^\infty}{\cong} M_1 \cup_{h'} M_2$.

We have the following example of pasting:

Let M_1 and M_2 be connected (oriented) n-manifolds and $f_i: D^n \hookrightarrow M_i$ $(i=1,2)$ be embeddings (f_1 orientation preserving, f_2 orientation reversing). Define

$$M_1 \# M_2 = (M_1 - \text{int } f_1(D^n)) \bigcup_{f_2 \circ f_1^{-1}|S^{n-1}} (M_2 - \text{int } f_2(D^n)).$$

$M_1 \# M_2$ is called the connected sum of M_1 and M_2. In the case M_1 and M_2 are oriented f_1 must preserve orientation and f_2 must reverse orientation to have a consistent orientation on M.

<u>Theorem 0.3.</u> Any two embeddings of D^n into the interior of a connected manifold M^n are isotopic (possibly after reversing orientations of one if necessary).

This theorem shows $\#$ is a well defined operation: i.e. $\#$ is independent of the choice of f_i.

Let N be a smooth manifold and define:

Diff (N) = diffeomorphism group of N

$\mathrm{Diff}_o(N)$ = identity component of Diff (N)

= diffeomorphisms isotopic to the id

Diff $(N)/\mathrm{Diff}_o(N)$ = isotopy classes of diffeomorphisms of N

$\mathrm{Diff}^+(N)$ = orientation preserving diffeomorphisms of N.

<u>Theorem 0.4.</u> $$\mathrm{Diff}\,(T^2)/\mathrm{Diff}_o(T^2) \cong GL(2,\mathbb{Z})$$

$$\mathrm{Diff}^+(T^2)/\mathrm{Diff}_o(T^2) \cong SL(2,\mathbb{Z})$$

A related result is

<u>Theorem 0.5.</u> An oriented simple closed curve in T^2 is uniquely determined (up to isotopy) by its homology class. Any class of the form $p(S^1 \times \{1\}) + q(\{1\} \times S^1)$ with $\gcd(p,q) = 1$ occurs. Hence up to automorphisms of T^2 there is only one curve.

<u>Dehn Surgery.</u>

<u>Definition 0.6.</u> Given M^3 a 3-manifold such that $T^2 \subseteq \partial M^3$, c a simple closed curve in T^2, define $M^3(c) = M^3 \cup_h (D^2 \times S^1)$ where h

is a diffeomorphism of T^2 onto $\partial D^2 \times S^1$ which takes c onto a meridian of $D^2 \times S^1$. (i.e. a curve in $T^2 = \partial D^2 \times S^1$ which is null homotopic in $D^2 \times S^1$.) M(c) is said to be obtained from M by Dehn surgery.

<u>Proposition 0.7.</u> $M^3(c)$ is well defined (up to diffeomorphism).

<u>Proof</u>: We can parametrize T^2 such that $c = S^1 \times \{1\} \subseteq S^1 \times S^1 = T^2$. Then $h(t,1) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}(t,1) = (t^a 1^b, t^c 1^d) = (t^a, t^c)$. $c = 0$ since $(t,1)$ gets mapped to $(t,1)$. Thus $h = \begin{pmatrix} \pm 1 & b \\ 0 & \pm 1 \end{pmatrix}$ which extends over the solid torus. By a previous theorem $M^3(c) \underset{C^\infty}{\cong} M^3 \cup_{id} (D^2 \times S^1)$ and hence is well defined.

More typically Dehn surgery is the following:

M^3 is the complement of a tubular neighborhood of a closed curve γ in some 3-manifold N^3. c = p (longitude) + q (meridian). This is called (p,q)-Dehn surgery on γ in N^3. Note that in general there are infinitely many possible choices of longitude in the boundary of a tubular neighborhood of γ. Therefore q is well defined only after making such a choice.

I. Definitions and Examples.

In this chapter we define and classify according to their Seifert invariant, Seifert and Generalized Seifert fibrations. A Seifert fibration over a orientable surface can be viewed also as the orbit space projection of an S^1-action on a 3-manifold. This is discussed in section 2. We extend the definition of the Euler number of an S^1-bundle to include Generalized Seifert fibrations. In section 4 the examples of lens spaces are described and used in Section 5 to describe (with proofs postponed) the classification of Seifert and Generalized Seifert fiberable (as opposed to fibered) manifolds. The final section 6 describes the basic algebraic topology of these manifolds.

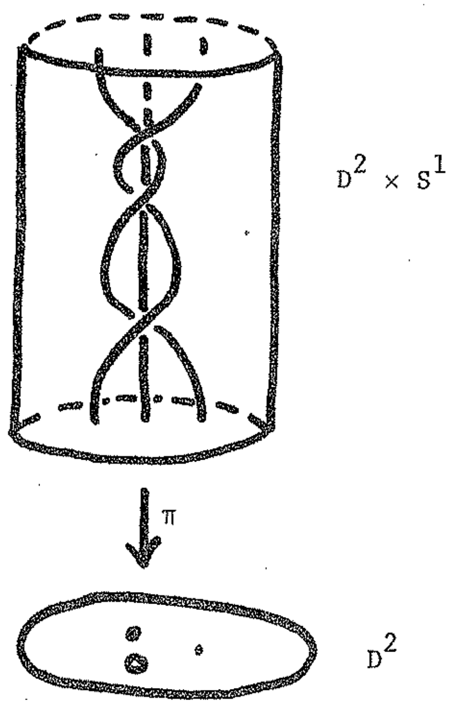
1. Seifert and Generalized Seifert Fibrations.

Definition 1.1. A G.S. (Generalized Seifert) fibration is a triple (M,F,π) (also denoted $M \xrightarrow{\pi} F$) where M is an oriented 3-manifold, F is a surface, oriented or unoriented, and $\pi: M \to F$ such that (M,F,π) is "almost" a locally trivial S^1-bundle. To be precise:
For every $x \in F$, there exists a D^2 neighborhood of x such that $\pi^{-1}(D^2) \cong D^2 \times S^1$ and

$$\pi: D^2 \times S^1 \to D^2 \text{ is defined by } (rt_1,t_2) \mapsto rt_1^p t_2^q$$

where $t_i \in S^1 = \{t \in \mathbb{C} \mid |t| = 1\}$, $r \in [0,1]$, $p,q \in \mathbb{Z}$ and the $\gcd(p,q) = 1$. Here the values of p and q depend on x. If for every $x \in F$, $p \neq 0$ then (M,F,π) is called a Seifert fibration.

To understand the local structure of a G.S. fibration we look at the above "local model" $\pi: D^2 \times S^1 \to D^2$. If $p \neq 0$ we can parametrize a typical fiber by $\pi^{-1}(rs) = (rs^{1/p}t^q, t^{-p})$, $s,t \in S^1$, $r \in (0,1]$. If we consider $D^2 \times S^1$ as $D^2 \times I$ with ends identified we get the following picture:



The center of the disc 0 lifts to the core circle of the solid torus and points in $D^2 - \{0\}$ lift to fibers that wrap p times around the core in the longitudinal direction and $-q$ times in the meridianal direction.

An alternative description is to consider $D^2 \times I$ fibered by lines $\{x\} \times I$. Form a solid torus $D^2 \times S^1$ by identifying the ends of the solid cylinder with a $2\pi q/p$ twist.

<u>Definition 1.2.</u> We call a fiber singular or exceptional if the value of p associated to this fiber is not equal to ± 1.

Proposition 1.3. If (M,F,π) is a G.S. fibration with F compact, the number of exceptional fibers is finite. (Note F is compact iff M is.)

Proof: For every $x \in F$ there exists a neighborhood D^2x such that $\pi^{-1}(D^2x)$ contains at most one exceptional fiber, namely $\pi^{-1}(x)$. Since F is compact we can cover it by finitely many such neighborhoods.

We now consider the case of a G.S. fibering (M,F,π) where F is closed oriented and connected. If we are given such a fibering, we can remove solid torus neighborhoods of suitable fibers of M and corresponding disc neighborhoods of F, to leave a genuine S^1-bundle over a connected orientable surface with boundary. Any such S^1-bundle is trivial, thus M is the result of Dehn surgery on some fibers of a trivial bundle $F \times S^1 \to F$. As already remarked, each Dehn surgery is determined by a suitable coprime integer pair.

To see exactly how this construction can be done, assume we are given data $(g; (\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ where $g \geq 0$, $\alpha_i,\beta_i \in \mathbb{Z}$, $\alpha_i \geq 0$, and $\gcd(\alpha_i,\beta_i) = 1$ $i = 1,\ldots,n$. Let F_o = oriented surface of genus g with n punctures: $F_o = F - (D_1^2 \cup \ldots \cup D_n^2)$ (F = closed surface of genus g). Define

$$M_o = F_o \times S^1$$
$$\partial M_o = S_1^1 \times S^1 \cup S_2^1 \times S^1 \cup \ldots \cup S_n^1 \times S^1$$

Let
$$R = F_o \times \{1\}$$
$$Q_i = R \cap (S_i^1 \times S^1) = S_i^1 \times \{1\} \quad \text{(oriented as a component of } -\partial R)$$
$$H_i = \{1\} \times S^1 \subseteq S_i^1 \times S^1$$

We have a trivial S^1-bundle (M_o, F_o, π). To construct a G.S. fibration from this bundle, paste a solid torus $T_i = D^2 \times S^1$ into the i^{th} boundary component $S_i^1 \times S^1$ in such a way that a meridian $M_i = S_i^1 \times \{1\} \subseteq \partial T_i$ satisfies the homology relation $M_i \sim \alpha_i Q_i + \beta_i H_i$ in the homology of ∂T_i.

If we let $L_i = \{1\} \times S^1 \subseteq \partial T_i$ and $M_i \sim \alpha_i Q_i + \beta_i H_i$ then $L_i \sim \alpha_i' Q_i + \beta_i' H_i$ for some $\begin{pmatrix} \alpha_i & \beta_i \\ \alpha_i' & \beta_i' \end{pmatrix} \varepsilon\ SL(2, \mathbb{Z})$. Therefore we can solve for H_i and Q_i in terms of M_i and L_i to get

$$-\alpha_i' M_i + \alpha_i L_i = H_i$$

$$\beta_i' M_i - \beta_i L_i = Q_i.$$

In T_i, $M_i \sim 0$, thus we have

$$\left.\begin{array}{l} H_i \sim \alpha_i L_i \\ Q_i \sim -\beta_i L_i \end{array}\right\} \text{ in the homology of } T_i.$$

This gives an alternate description of what α_i and β_i signify, i.e. α_i is the number of times H_i wraps around T_i and $-\beta_i$ the number of times Q_i wraps around T_i.

We denote the G.S. fibration constructed above by $M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ and call $\{g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)\}$ the Seifert invariant.

Definition 1.4. We say two G.S. fibrations (M,F,π), (M',F',π') are isomorphic iff there exist diffeomorphisms $f: F \to F'$ and $\tilde{f}: M \to M'$, with $\tilde{f}$ orientation preserving such that

$$\begin{array}{ccc} M & \xrightarrow{\tilde{f}} & M' \\ \pi\downarrow & & \downarrow\pi' \\ F & \xrightarrow{f} & F' \end{array} \quad \text{commutes}$$

We shall see that different Seifert invariants can result in isomorphic G.S. fibrations. The second part of the next theorem gives necessary and sufficient conditions on the Seifert invariant to yield isomorphic G.S. fibrations. The first part shows every G.S. fibration can be obtained by the above method (provided F is oriented).

Theorem 1.5. Let $M \xrightarrow{\pi} F$ be a G.S. fibration with F closed connected and oriented. Then

a) $(M \xrightarrow{\pi} F) \cong M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ for some $g,\alpha_i,\beta_i \in \mathbb{Z}$

b) $M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)) \cong M(g';(\alpha_1',\beta_1'),\ldots,(\alpha_m',\beta_m'))$ iff;

i) $g = g'$

ii) Disregarding any β_i/α_i and β_j'/α_j' which are integers ($\neq \infty$), the remaining β_i/α_i (mod 1) are a permutation of the remaining β_j'/α_j' (mod 1)

iii) $\sum_{i=1}^{n} \beta_i/\alpha_i = \sum_{j=1}^{m} \beta_j'/\alpha_j'$ where here we use the convention $1/0 = -1/0 = \infty$, $\infty + x = \infty$ for every $x \in \mathbb{R} \cup \{\infty\}$.

Equivalently, the following collection of operations can be done to a Seifert invariant without changing the corresponding G.S. fibration up to isomorphism:

I) Add or delete any Seifert pair $(\alpha,\beta) = (1,0)$

II) Replace any $(0,\pm 1)$ by $(0,\mp 1)$

III) Replace each (α_i,β_i) by $(\alpha_i,\beta_i + K_i\alpha_i)$ provided $\sum K_i = 0$.

<u>Example</u>. $M(0;(2,1),(3,2),(5,-6)) \cong M(0;(2,-1),(3,2),(5,-1))$

$\cong M(0;(1,0),(2,1),(3,2),(5,-6))$

$\cong M(0;(1,-2),(2,1),(3,2),(5,4))$.

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<u>Proof of Theorem 1.5</u>. a) Choose points $p_1,\ldots,p_n \varepsilon F$ such that $\{\pi^{-1}(p_1),\ldots,\pi^{-1}(p_n)\}$ includes all exceptional fibers. Let $D_1,\ldots,D_n$ be disjoint disc neighborhoods of the p_i. Then $T_i = \pi^{-1}(D_i)$ are disjoint solid torus neighborhoods of the $\pi^{-1}(p_i)$. Define $M_o = M - \mathrm{int}(T_1 \cup \ldots \cup T_n)$ and $F_o = F - \mathrm{int}(D_1 \cup \ldots \cup D_n)$. $(M_o,F_o,\pi|_{F_o})$ is a genuine S^1-bundle and therefore trivial. Hence we can find a section $s: F_o \to M_o$. Define $R = s(F_o) \subset M_o$, $Q_i = R \cap \partial T_i$ and let H_i be a non-singular fiber in ∂T_i. Let α_i be the multiple of the generator of $H_1(T_i)$ that H_i represents and $-\beta_i$ the multiple of the generator of $H_1(T_i)$ that Q_i represents. Then $(M,F,\pi) \cong M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ where $g = \mathrm{genus}(F)$, thus proving a).

b) By part a), we can assume that given any G.S. fibration $M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ that (α_i,β_i) were obtained by the method used in the proof of part a). If we had choosen extra points p_i, i.e. points whose fibers are not exceptional, the result would have been to introduce pairs $(1,0)$ into the Seifert invariant. We also made an arbitrary choice when defining the section $s: F_o \to M_o$. With respect to a suitable trivialization $M_o \cong F_o \times S^1$, the section s is given by $s(x) = (x,1)$, and then any other section s' has the form $s' \colon x \mapsto (x,\phi(x))$ where $\phi \colon F_o \to S^1$. We can change s' by an isotopy without changing the corresponding values of (α_i,β_i). Thus we are concerned only with the homotopy class of ϕ. We claim:

> If we denote $\phi|\partial F_o \in [\partial F_o, S^1] = \mathbb{Z}^n$ by $(q_1,\ldots,q_n)$ where $q_i = \deg \phi|s_i'$, then $(q_1,\ldots,q_n)$ occurs for some ϕ iff $q_1 + \ldots + q_n = 0$.

To prove this recall $H^1(X,\mathbb{Z}) = [X,S^1]$ and note the following exact sequences:

$$\begin{array}{ccccc}
H^1(F_o) & \longrightarrow & H^1(\partial F_o) & \xrightarrow{\alpha} & H^2(F_o,\partial F_o) \\
\| & & \| & & \| \\
H_1(F_o,\partial F_o) & \longrightarrow & H_o(\partial F_o) & \longrightarrow & H_o(F_o) \\
 & & \| & & \| \\
 & & \mathbb{Z}^n & & \mathbb{Z}
\end{array}$$

α is the map $\alpha(Z_1,\ldots,Z_n) = Z_1 + \ldots + Z_n$. Therefore $(q_1,\ldots,q_n)$ occurs iff it pulls back to $H^1(F_o)$ iff $q_1 + \ldots + q_n = 0$.

We see from the claim that in choosing s' insead of s, Q_j is replaced by $Q_j + q_j H$ which winds $-\beta_j + q_j\alpha$ times around the solid torus T_j. Hence we can replace each β_j by $\beta_j - q_j\alpha_j$ provided $\sum q_j = 0$.

Note the non-uniqueness of the Seifert invariant is due to the arbitrary choice of a section $s: F_o \to M_o$. If we calculate the Seifert invariant with respect to a fixed section, two pairs, consisting each of a G.S. fibration plus a section to the fibration outside a finite collection of fibers, are isomorphic iff their Seifert invariants are equal (up to permutation of pairs).

<u>Corollary 1.6</u>. The Seifert invariant of a Seifert fibration has unique normal form (up to permutation of indices)

$$M(g;(1,\beta_o),(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)), \quad 0 < \beta_i < \alpha_i, \quad i = 1,\ldots,n.$$

If (M,F,π) is a G.S. fibration which is not a Seifert fibration, we can uniquely represent it (up to permutation of indices) as:

$$M(g;(0,1),\ldots,(0,1),(\alpha_1,\beta_1),\ldots,(\alpha_m,\beta_m)), \quad 0 < \beta_i < \alpha_i, \; i = 1,\ldots,m.$$

<u>Definition 1.7</u>. $e(M \to F) = -\sum \alpha_i/\beta_i$ is called the Euler number of the G.S. fibration $M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$. $e(M \to F) \neq \infty$ iff (M,F,π) is a Seifert fibration.

Note that the Seifert invariant is an invariant of the oriented manifold M with its fibered structure, it does not depend on the orientation of the base F. For if we reverse the orientation of F, we must reverse the orientation of the fibers also, to keep the orientation of M fixed. Thus both Q_i and H_i are reversed, and the homology relation $\alpha_i Q_i + \beta_i H_i \sim 0$ in T_i, which determines (α_i,β_i), is unchanged. This can also be interpreted as saying that there exists a fiber preserving diffeomorphism $f: M \to M$, preserving orientation of M, such that the induced map $F \to F$ reverses orientation.

Exercise: Show f can be chosen even as an involution ($f^2 = id$).

Note also that reversing the orientation of M reverses the sign of either Q_i or H_i, so β_i/α_i gets replaced by $-\beta_i/\alpha_i$. Thus we have:

Corollary 1.7. If $M = M(g;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ then $-M = M(g;(\alpha_1,-\beta_1),\dots,(\alpha_n,-\beta_n))$. In particular $e(M \to F) = -e(-M \to F)$ ($-M$ means M with reversed orientation).

We now consider the case where F is non-orientable. Then $F = F_1 \# F_2$ where F_1 is an orientable surface and $F_2 = \mathbb{RP}^2$ or $F_2 = \mathbb{RP}^2 \# \mathbb{RP}^2$. By homogeneity of manifolds, we can assume the singular fibers of (M,F,π) lie only over points of F_1. Therefore, over F_2 we have a genuine S^1-bundle with oriented total space.

We now introduce Seifert invariants as before:

a) Remove tubular neighborhoods of the singular fibers (and possibly some non-singular fibers). This gives $M_o \to F_o \# F_2$, a genuine S^1-bundle where $F_o = F_1 - (D_1^2 \cup \dots \cup D_n^2)$, $M_o = M - (T_1 \cup \dots \cup T_n)$.

b) Choose a section $R \subset M_o$ to the fibration and use this to compute the Seifert pairs (α_i,β_i). This gives a Seifert invariant $(g;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ where $g < 0$ is the genus of F (we use negative genus for nonorientable surfaces, i.e. $F = \mathbb{RP}^2 \# \dots \# \mathbb{RP}^2$, $|g|$ times). As before we have

Theorem 1.8. Let $M \xrightarrow{\pi} F$ be a G.S. fibration with F closed connected and unorientable. Then

a) $(M \xrightarrow{\pi} F) \cong M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ for some $g,\alpha_i,\beta_i \in \mathbb{Z}$ with $g < 0$

b) same as the orientable case, i.e. we can change any β_i/α_i by an integer provided we keep $-\sum \beta_i/\alpha_i = e(M \to F)$ fixed. We can add or delete pairs $(\alpha_i,\beta_i) = (1,0)$.

Proof: Let $F = F' \# \mathbb{R}P^2$ where $F' = F_1$ or $F' = F_1 \# \mathbb{R}P^2$. Then

$$\begin{aligned} F &= (F' - \mathrm{int}(D^2)) \cup_{S^1} (\mathbb{R}P^2 - \mathrm{int}(D^2)) \\ &= (F' - \mathrm{int}(D^2)) \cup_{S^1} (Mb) \qquad (Mb = \text{Moebuis band}). \end{aligned}$$

We need:

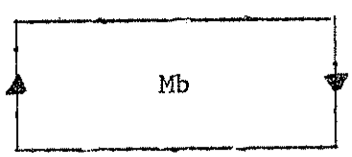
Lemma 1.9. Suppose $E \xrightarrow{\pi} Mb$ is a fibration with fiber S^1 and oriented total space. Then

i) There is only one such E up to isomorphism namely $E = T^1 Mb =$ unit tangent bundle.

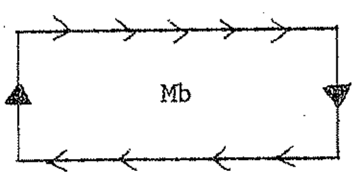
ii) There is exactly one section up to isotopy of $E|\partial(Mb)$ which extends over E.

Given this lemma, there is a canonical way of cutting out $\pi^{-1}(Mb)$ in M and replacing it by $D^2 \times S^1$, to get a G.S. fibration over F'. The proof thus reduces to the case of F orientable.

Proof of Lemma 1.9. i) Classifying bundles $E \xrightarrow{\Pi} Mb$ with S^1 fiber and orientable total space is equivalent to classifying, up to homotopy, orientation reversing diffeomorphisms $h: S^1 \to S^1$. There is only one such h. Thus $E \to Mb$ is unique, and $E \cong T^1Mb$ since T^1Mb is such a bundle. ii) A section of T^1Mb is a unit vector field on Mb. Call the section on ∂Mb that is parallel to ∂Mb the trivial section. We claim any section $r: Mb \to T^1Mb$ is such that $r|\partial Mb$ is isotopic to the trivial section. To see this, choose a very narrow Móebius band neighborhood U of the core circle. As you traverse the boundary of U, any "rotation" of the vector field gets canceled as you "come around the second time" and therefore $r|\partial U$ is isotopic to the trivial section. Since $Mb - U =$ collar $= \partial Mb \times I$, this gives an isotopy of $r|\partial Mb$ to the trivial section (see below).

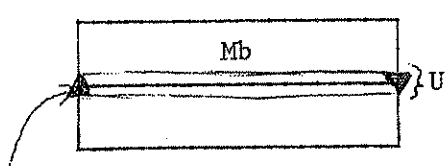


This is the Moebius band.



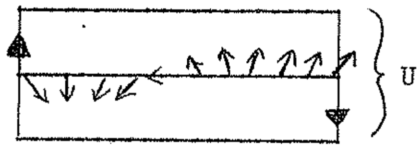
This is the trivial section on ∂Mb.

Given any section r of Mb, choose U.

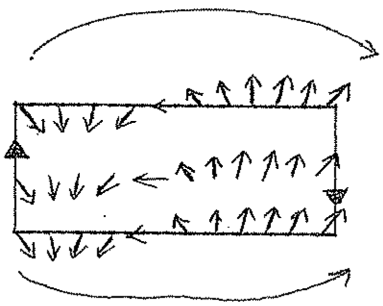


core circle

On the core circle the section looks like:



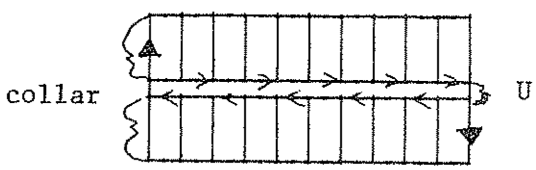
By continuity of r, on U the section looks like:



as you go around here the rotation is α ;

as you go around here the rotation is $-\alpha$.

Thus this section is isotopic to the trivial section



Now using the collar extend this isotopy to give an isotopy of $r|\partial Mb$ to the trivial section.

2. Seifert Fibrations as S^1-Actions.

Observe, if (M,F,π) is a Seifert fibration with M closed and oriented and F oriented, by the way M is constructed, we can put an S^1 action on M. The orbits of this action are the fibers of π. If (M,F,π) is a G.S. fibration with F oriented, then there is an S^1 action on M such that each fiber of π is either

a) an orbit if the fiber is non-singular or singular with $\alpha \neq 0$

b) a component of a fixed point set if it is a singular fiber with $\alpha = 0$.

The converse is also true.

Theorem 2.1. The classification of G.S. fibrations with F orientable is equivalent to the classification of effective S^1 actions on closed oriented 3-manifolds.

Before proving this theorem we need some definitions and results from the theory group actions on a manifold. A reference for these results is G. Bredon: Introduction to Compact Transformation Groups.

Definition. Let M be a smooth manifold and G a compact Lie group. A *smooth G-action* on M is a C^∞ map $G \times M \to M$, $(g,x) \mapsto gx$ satisfying

i) $1x = x$ for every $x \in M$

ii) $g_1(g_2x) = (g_1g_2)x$ for every $g_1, g_2 \in G$, for every $x \in M$.

This action is <u>effective</u> if $gx = x$ for every $x \in M$, then $g = 1$. This action is <u>fixed point free</u> if for every $x \in M$, there exists $g \in G$ such that $gx \neq x$. This action is <u>free</u> if for every $x \in M$ and for every $g \neq 1$, $gx \neq x$. We define the <u>orbit space</u> $M/G = M/\{x \equiv gx\}$, the <u>orbit</u> $Gx = \{gx \mid g \in G\}$ and we define the <u>isotropy subgroup</u> $G_x = \{g \in G \mid gx = x\}$.

<u>Lemma 2.2</u>. G_x is a closed subgroup of G. $G/G_x \cong Gx$ where this diffeomorphism is G-equivariant and given by $gG_x \mapsto gx$.

<u>Example</u>: 1) Let H be a compact Lie group and $\rho: H \to GL(n)$ be a representation. Then H acts on $\mathbb{R}^n$ by $h \cdot x = \rho(h)x$, $x \in \mathbb{R}^n$, $h \in H$.

2) Suppose $H \subseteq G$ is a closed subgroup and ρ is as above. We define $G \times_H \mathbb{R}^n = G \times \mathbb{R}^n / H$ where H acts on $G \times \mathbb{R}^n$ by $h(g,x) = (gh^{-1}, \rho(h)x))$.

<u>Theorem 2.3</u>. Let G,H be as in example 2). Then $G \times_H \mathbb{R}^n \to G/H$ given by $[g,v] \mapsto gH$ is a vector bundle with fiber $\mathbb{R}^n$. It has a natural G action given by $g_1[g,v] = [g_1 g, v]$.

<u>Theorem 2.4</u>. (Slice Theorem) Let G be a compact Lie group and $G \times M \to M$ a smooth action of G. Then

1) $Gx \subseteq M$ is a smooth submanifold

2) G_x acts on $V_x = \nu_x(Gx)$ by a representation $\rho_x: G_x \to GL(V_x)$ ($\nu_x(Gx)$ = normal bundle of Gx in M at x), called the "slice representation"

3) $G \times_{G_x} V_x$ is G-equivariantly diffeomorphic to a neighborhood of $Gx \subseteq M$ by a diffeomorphism which takes the zero section

$G/G_x = \{[g,0]\}$ to Gx.

4) After choosing an invariant Euclidean metric on V_x we can assume $\rho_x\colon G_x \to O(V_x)$.

We now return to Theorem 2.1. To prove this theorem, it suffices to show any effective S^1 action on a closed oriented 3-manifold M yields a G.S. fibration. By part 3 of the Slice theorem, if we know all the possible slice representations of the isotropy subgroups of S^1, we know what the orbits look like locally. We must show that locally these orbits look like a local model as in the definition of a G.S. fibration.

In our case $G = S^1$ and the possible isotropy subgroups are $G_x = \{1\}, \mathbb{Z}/n, S^1$.

<u>Case 1</u>. $G_x = \{1\}$

Then $G \times_{G_x} V = G \times V = S^1 \times \mathbb{R}^2$ since $\dim V = 2$.

<u>Case 2</u>. $G_x = \mathbb{Z}/n$

As in case 1, since $\dim G \times_{G_x} V = 3$ and G_x is discrete, we have $\dim V = 2$. For the action of G to be effective $\rho\colon G_x \to O(V)$ must be injective. If $n > 2$ the only possibility for ρ is a generator of $\mathbb{Z}/n$ goes to a rotation by $2\pi q/n$ where $\gcd(q,n) = 1$. If $n = 2$ we have the additional possibility, a generator goes to $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$. In this case $G \times_{G_x} V$ is non-orientable and thus cannot occur. Thus $G \times_{G_x} V \cong (S^1 \times \mathbb{R}^2)/(\mathbb{Z}/n)$ which is again a standard model.

<u>Case 3</u>. $G_x = S^1$

In this case $\dim V = 3$. Again, for the action to be effective

$\rho\colon S^1 \to O(V)$ must be injective. There is only one such representation, namely writing $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}^1$, S^1 rotates $\mathbb{R}^2$ and fixes $\mathbb{R}^1$. Thus $G \times_{G_x} V = S^1 \times_{S^1} \mathbb{R}^3 \cong \mathbb{R}^3$ as S^1 manifolds. The fixed point set is thus a closed one dimensional submanifold of M so a component of the fixed point set looks like a $(0,1)$ fiber in a G.S. fibration.

Thus we have shown that given any orientable closed 3 manifold M with an effective S^1 action and orbit space M/S^1, then the non-fixed orbits and the fixed point components induce a G.S. fibration on M.

<u>Proposition 2.5</u>. Let M be a closed orientable effective S^1 manifold with Seifert invariant $(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ and $\mathbb{Z}/a \subseteq S^1$. Then $M/(\mathbb{Z}/a)$ is a $(S^1/(\mathbb{Z}/a))$ manifold and its Seifert invariant can be written $(g;(\alpha_1',\beta_1'),\ldots,(\alpha_n',\beta_n'))$ where $\beta_j'/\alpha_j' = a\beta_j/\alpha_j$. In particular $e(M/(\mathbb{Z}/a) \to M/S^1) = a\cdot e(M \to M/S^1)$.

<u>Proof</u>: By checking the local structure, $M/(\mathbb{Z}/a)$ is easily seen to be a 3-manifold. Remove from M tubular neighborhoods of a suitable collection of orbits. In M – {tubular neighborhoods} choose a section R which gives the stated Seifert invariant for M. The image of R in $M/(\mathbb{Z}/a)$ is still a section and this section gives the desired Seifert invariant for $M/(\mathbb{Z}/a)$.

3. Euler Number.

There are several equivalent ways of defining the Euler number for a genuine S^1-bundle. We list three:

1) An obstruction to finding a cross section.
2) A bundle is classified by an element of $[F,BS^1] = [F,K(\mathbb{Z},2)] = H^2(F;\mathbb{Z})$. If F is a closed surface $H^2(F;\mathbb{Z}) = \mathbb{Z}$ and the $\alpha \in H^2(F,Z)$ classifying the S^1 bundle is the Euler number.
3) "Fill in the circle fibers" to get a D^2 bundle $E \to F$ with $\partial E = M$ $(E = M \times_{S^1} D^2 = (M \times D^2)/S^1)$. We have the zero section $F \hookrightarrow E$. Then $e(M \to F) = [F]\cdot[F]$ (self intersection number).

One can show that each of these ways has an extension to Seifert fibrations and they all give the same result. Our definition of $e(M \to F) = -\sum \alpha_i/\beta_i$ corresponds to the first definition. In this section we will give a definition of $e(M \to F)$ corresponding to 3), and it will be used in the proof of a theorem. To 2) we remark without proof that if $S^1_{(0)}$ is the rationalized circle, then to any Seifert bundle $M \to F$ is associated a genuine fibration over F with fiber $S^1_{(0)}$, classified by an element of $[F,BS^1_{(0)}] = [F,K(\mathbb{Q},2)] = H^2(F;\mathbb{Q})$, and this too is our Euler number.

Before stating and proving the main theorem of this section we need a proposition and in particular the corollary following.

<u>Proposition 3.1.</u> Let $M \overset{\pi}{\to} F$ be a G.S. fibration and $p\colon F' \to F$ a covering with degree $(p) = d$. We can form the pullback bundle $p^*M \to F'$ where this diagram commutes:

$$\begin{array}{ccc} p^*M & \xrightarrow{\bar{p}} & M \\ \pi' \downarrow & & \downarrow \pi \\ F' & \xrightarrow{p} & F \end{array}$$

Then

i) $p^*M \to F'$ is a G.S. fibration

ii) If $M \overset{\pi}{\to} F \cong M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ then $p^*M \overset{\pi'}{\to} F' \cong M(g';d(\alpha_1,\beta_1),\ldots,d(\alpha_n,\beta_n))$ where $d(\alpha_i,\beta_i) = (\alpha_i,\beta_i),\ldots,(\alpha_i,\beta_i)$ (d times) and $g' = \text{genus } F'$.

<u>Proof</u>. $p^*M = \{(x,y) \in M \times F' | \; p(y) = \pi(x)\}$. Choose a section R to $M \overset{\pi}{\to} F$ outside of a collection of tubular neighborhoods of suitable fibers. Then $\{(x,y) \in R \times F' | \; \pi(x) = p(y)\}$ is a similar section in p^*M leading to the desired Seifert invariant.

<u>Corollary 3.2</u>. Let $M \overset{\pi}{\to} F$ be a G.S. fibration with Seifert invariant $(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ with $g < 0$, i.e. F is unorientable. Let $\bar{F} \overset{p}{\to} F$ be the orientation double cover of F. We form the pullback bundle $\bar{M} \overset{\bar{\pi}}{\to} \bar{F}$. Then:

i) $\bar{M} \overset{\bar{\pi}}{\to} \bar{F}$ is a G.S. fibration.

ii) $\bar{M} \overset{\bar{\pi}}{\to} \bar{F} \cong M(|g| - 1; (\alpha_1,\beta_1),(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n),(\alpha_n,\beta_n))$.

In particular $e(\bar{M} \to \bar{F}) = 2e(M \to F)$.

We now state the main theorem of this section.

<u>Theorem 3.3</u>. Let $M_1 \xrightarrow{\pi_1} F_1$ and $M_2 \xrightarrow{\pi_2} F_2$ be two Seifert fibrations. Assume there exists a map $\tilde{g}: M_1 \to M_2$ such that the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{\tilde{g}} & M_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ F_1 & \xrightarrow{g} & F_2 \end{array}$$

commutes and degree $(g) = b$, degree $(\tilde{g}|_{\text{fiber}}) = f$ and thus degree $(\tilde{g}) = bf$. Then $e(M_1 \to F_1) = (b/f)e(M_2 \to F_2)$. (Note: The pair (b,f) is determined only up to sign, but b/f is well defined.)

We leave as an exercise the

Remark: The theorem is valid also for G.S. fibrations.

In proving this theorem, we shall also show that $e(M \to F)$ is equal to the self intersection number of the zero section of the corresponding "disc bundle," which we now define.

Given a Seifert fibration (M,F,π), let $C(\pi)$ be the mapping cylinder, i.e. $C(\pi) = M \times [0,1] \cup_t F$ where $t\colon M \times \{1\} \to F$ is $t(x,1) = \pi(x)$. π induces a mapping $\bar{\pi}\colon C(\pi) \to F$ whose fibers are the cones over the fibers of π. This is a "Seifert disc bundle" over F, and $M = \partial(C(\pi))$ is the corresponding circle bundle. We have $F \hookrightarrow C(\pi)$ as the zero section. $C(\pi)$ is a 4-manifold except at points $p \in F$ over which the singular orbits of M lie.

Definition 3.4. The pair (X,Y) is an R-homology manifold pair of dimension n iff (X,Y) is a relative C.W. complex and

$$\tilde{H}_i(X, X - \{p\}; R) = \begin{cases} R & i = n \\ 0 & i \neq n \end{cases}$$

for all $p \in X - Y$.

Claim. $(C(\pi),M)$ is a $\mathbb{Q}$ homology manifold pair.

Exercise. If G is a finite group with an orientation preserving action on a manifold N, then N/G is a $\mathbb{Q}$ homology manifold.

Proof of Claim. Using the above exercise, and since being an R-homology manifold is a local condition, it suffices to show that if $p \in F$ in $C(\pi)$ is a singular point, then p has a neighborhood homeomorphic to $\mathbb{R}^4/G$ for some finite G. Let p be a singular point and U a neighborhood of p. $\pi^{-1}(U) \overset{\pi}{\to} U$ "looks like" $S^1 \times_{\mathbb{Z}_\alpha} \mathbb{R}^2 \to \mathbb{R}^2/(\mathbb{Z}/\alpha)$ where $(\mathbb{Z}/\alpha)$ acts diagonally on $S^1 \times \mathbb{R}^2$, i.e. by right multiplication on S^1 and by some rotation on $\mathbb{R}^2$. The corresponding neighborhood of $\pi^{-1}(p)$ in $C(\pi)$ is $D^2 \times_{(\mathbb{Z}/\alpha)} \mathbb{R}^2 \cong (D^2 \times \mathbb{R}^2) / (\mathbb{Z}/\alpha)$.

Now the standard treatment of R-orientation, fundamental classes and Poincaré-Lefschetz duality (as for instance in Spanier: Algebraic Topology) carries through for R-homology manifold pairs (X,Y) with X compact and Y closed. Therefore $(C(\pi),M)$ satisfies Poincaré-Lefschetz duality with $\mathbb{Q}$ coefficients. Precisely, the sum of the top dimensional simplices in a subdivision of $C(\pi)$ defines a fundamental class $[C(\pi)] \in H_4(C(\pi),M;\mathbb{Q})$, and the maps

$$D: H^q(C(\pi);\mathbb{Q}) \to H_{4-q}(C(\pi),M;\mathbb{Q}) \quad \text{and}$$
$$D: H^q(C(\pi),M;\mathbb{Q}) \to H_{4-q}(C(\pi);\mathbb{Q})$$

defined by $D(\alpha) = \alpha \cap [C(\pi)]$ are isomorphisms.

In analogy to the case of a genuine S^1-bundle we now define

$e'(M \to F) = D(D^{-1}([F]) \cup D^{-1}([F]))$ where $[F] \in H_2(C(\pi);\mathbb{Q})$ is the homology class represented by $F \subseteq C(\pi)$. Denoting $D^{-1}(F) = \eta \in H^2(C(\pi);\mathbb{Q})$ we have

$$\begin{aligned} e'(M \to F) &= (\eta \cup \eta) \cap [C(\pi))] \\ &= \eta \cap (\eta \cap [C(\pi)]) \\ &= \eta \cap [F]. \end{aligned}$$

The proof of Theorem 3.3 is divided into two steps. The first step is to show $e'(M_1 \to F_1) = (b/f)\, e'(M_2 \to F_2)$. The second step is to show $e'(M \to F) = e(M \to F)$.

It may appear that we are implicitly assuming F is orientable above. However, if we take our coefficients $\mathbb{Q}$ in $H_2(C(\pi);\mathbb{Q})$ and $H^2(C(\pi),M;\mathbb{Q})$ to be the local coefficient system on $C(\pi)$ which pulls back from the orientation system on F (but take untwisted coefficients for H_4 and H^4), then our definition of $e'(M \to F)$, and the subsequent analysis, applies also for F unorientable. In the following proof we therefore implicitly assume these local coefficients are being used where necessary. The reader who prefers to avoid local coefficients can instead deduce the theorem in general from the special case of oriented base surfaces using corollary 3.2.

Proof of Theorem 3.3. (Step 1) We have

$$\begin{array}{ccc} M_1 & \xrightarrow{\tilde{g}} & M_2 \\ \downarrow & & \downarrow \\ F_1 & \xrightarrow{g} & F_2 \end{array} ,$$

this induces $G\colon C(\pi_1) \to C(\pi_2)$ with $G|M_1 = \tilde{g}$ and $G|F_1 = g$. Then degree (G) = degree $(\tilde{g}) = bf$. Let $\eta_i \in H^2(C(\pi_i);\mathbb{Q})$ $(i = 1,2)$ be as above. Then $e'(M_i \to F_i) = \eta_i \cap [F_i]$. $H^2(C(\pi_i);\mathbb{Q}) = H^2(F_i;\mathbb{Q}) = \mathbb{Q}$, so $G^*\eta_2$ is some multiple $k\eta_1$ of η_1. Thus

$$G^*\eta_2 \cap [C(\pi_1)] = k\eta_1 \cap [C(\pi_1)] = k[F_1]$$

$$G_*((G^*\eta_2) \cap [C(\pi_1)]) = G_*k[F_1] = bk[F_2].$$

On the other hand

$$\begin{aligned} G_*((G^*\eta_2) \cap [C(\pi_1)]) &= \eta_2 \cap G_*[C(\pi_1)] \\ &= bf\, \eta_2 \cap [C(\pi_2)] \\ &= bf[F_2] \end{aligned}$$

and hence $k = f$. Therefore $G^*\eta_2 = f\eta_1$, so

$$\begin{aligned} e'(M_1 \to F_1) &= \eta_1 \cap [F_1] \\ &= (1/f)G^*(\eta_2) \cap [F_1] \\ &= (1/f)G_*(G^*\eta_2 \cap [F_1]) \\ &= (1/f)(\eta_2 \cap G_*[F_1]) \\ &= (b/f)(\eta_2 \cap [F_2]) \\ &= (b/f)e'(M_2 \to F_2). \end{aligned}$$

(Step 2) We want to show that for any Seifert manifold $e(M \to F) = e'(M \to F)$. First assume F is oriented. Then M is a fixed point free

S^1-manifold and $M \cong M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$. Let a be a multiple of $\mathrm{lcm}(\alpha_1,\ldots,\alpha_n)$ and define $M' = M/(\mathbb{Z}/a)$. Then $M' = M(g;\ (1,a\beta_1/\alpha_1),\ldots,(1,a\beta_n/\alpha_n)) = M(g;\ (1,\ \sum a\beta_i/\alpha_i))$. Thus M' is a genuine S^1-bundle.

<u>Exercise</u>: For a genuine S^1-bundle $e' = e$.

With this exercise we have

$$\begin{aligned}(1/a)e(M \to F) &= e(M' \to F) && \text{(Proposition 3.1)}\\ &= e'(M' \to F) && \text{(exercise)}\\ &= (1/a)e'(M \to F) && \text{(Step 1).}\end{aligned}$$

Now if F is not oriented, let $\bar{M} \to \bar{F}$ be as constructed in Corollary 3.2. Then

$$\begin{aligned}e(M \to F) &= (1/2)e(\bar{M} \to \bar{F}) && \text{(Corollary 3.2)}\\ &= (1/2)e'(\bar{M} \to \bar{F}) && \text{(above)}\\ &= e'(M \to F) && \text{(Step 1).}\end{aligned}$$

4. Lens Spaces as Seifert Manifolds.

In this section we define lens spaces and show they are examples of Seifert fibrations. As we shall see in the next section, this allows us to give a classification of Seifert structures on a 3-manifold.

Definition 4.1. Let $S^3 = \{(Z_1,Z_2) \in \mathbb{C}^2 |\ |Z_1|^2 + |Z_2|^2 = 1\}$. $\mathbb{Z}/p$ acts freely on S^3 by $e^{2\pi ip}(Z_1,Z_2) = (e^{2\pi i/p}Z_1, e^{2\pi iq/p}Z_2)$ where $\gcd(p,q) = 1$. Define the lens space $L(p,q) = S^3/(\mathbb{Z}/p)$.

$L(p,q)$ has the following properties:

1) $S^3 \to L(p,q)$ is the universal cover.
2) Since the covering transformation group is $\mathbb{Z}/p$

   $\pi_1(L(p,q)) = \mathbb{Z}/p$
3) By elementary algebraic topology we get

   $H_1(L(p,q)) = \mathbb{Z}/p$

   $H_2(L(p,q)) = 0$

   $H_3(L(p,q)) = \mathbb{Z}$.
4) $L(p,q) \cong L(p,q')$ if $q \equiv q' \pmod p$
5) $L(p,q) \cong L(p,q')$ if $qq' \equiv 1 \pmod p$

   Proof: $e^{2\pi iq'/p}$ is a generator of $\mathbb{Z}/p$ and in $L(p,q)$ it takes $(Z_1,Z_2) \mapsto (e^{2\pi iq'/p}Z_1, e^{2\pi iqq'/p}Z_2) = (e^{2\pi iq'/p}Z_1, e^{2\pi i/p}Z_2)$. Thus by exchanging Z_1 and Z_2 the (p,q) action becomes the (p,q') action. Therefore $L(p,q) \cong L(p,q')$.
6) $L(p,q) \cong -L(p,-q)$

   Proof: The map $(Z_1,Z_2) \mapsto (Z_1,\bar{Z}_2)$ induces $L(p,q) \xrightarrow{\cong} -L(p,-q)$.

7) $L(p,q) \simeq L(p,q')$ (homotopy equivalent preserving orientation) if qq' is a square mod p.

Proof. See exercise 1.

Theorem 4.2. $L(p,q) \overset{\sim}{=} L(p,q')$ iff $q \equiv (q')^{\pm 1} \pmod p$

$L(p,q) \simeq L(p,q')$ iff $qq' \equiv$ square (mod p).

Here both the diffeomorphism and the homotopy equivalence are orientation preserving.

Proof: See J. H. C. Whitehead, "On incidence matrices, nuclei and homotopy types," Ann. of Math. vol. 42, 1941.

$T^2 = S^1 \times S^1$ acts on S^3 by $(t_1,t_2)(Z_1,Z_2) = (t_1Z_1, t_2Z_2)$. The above $\mathbb{Z}/p$ action is a subaction of T^2 on S^3. Thus $T^2/(\mathbb{Z}/p)$ acts on $S^3/(\mathbb{Z}/p) \simeq L(p,q)$. $T^2/(\mathbb{Z}/p) \overset{\sim}{=} T^2$. There are many S^1 subgroups of $T^2/(\mathbb{Z}/p)$ giving effective actions on $L(p,q)$, hence there are many Seifert and G.S. fibrations on $L(p,q)$.

Theorem 4.3. $L(p,q) \overset{\sim}{=} D^2 \times S^1 \underset{\begin{pmatrix} -q & r \\ p & s \end{pmatrix}}{\cup} D^2 \times S^1$ if $\det\begin{pmatrix} -q & r \\ p & s \end{pmatrix} = -1$.

Remark. If we define $L(p,q) = L(-p,-q)$ if $p < 0$

$L(1,0) = S^3$

$L(0,1) = S^1 \times S^2$,

then the theorem remains true with these conventions.

Proof of Theorem 4.3. Let $S^3 = \{(Z_1,Z_2) \mid |Z_1|^2 + |Z_2|^2 = 2\}$. Then $S^3 = U_1 \cup U_2$ where $U_1 = \{(Z_1,Z_2) \in S^3 \mid |Z_2|^2 \geq 1\}$ and

$U_2 = \{(Z_1,Z_2) \in S^3 | \ |Z_2|^2 \leq 1\}$. $U_1 \cong (D^2 \times S^1)_1$ where the diffeomorphism is given by $(Z_1,Z_2) \mapsto (Z_1, Z_2/\|Z_2\|)$. Similarly $U_2 \cong (D^2 \times S^1)_2$ by $(Z_1,Z_2) \mapsto (Z_2, Z_1/\|Z_1\|)$. Thus the pasting map $\partial(D^2 \times S^1)_1 = S^1 \times S^1 \to S^1 \times S^1 = \partial(D^2 \times S^1)_2$ is $(Z_1,Z_2) \mapsto (Z_2,Z_1)$ i.e. given by $\begin{pmatrix}0&1\\1&0\end{pmatrix}$. Now if we factor by the $\mathbb{Z}/p$ action we have $(D^2 \times S^1)_1/(\mathbb{Z}/p) \overset{\cong}{\to} D^2 \times S^1$ by $[Z_1,Z_2] \mapsto (Z_2^{-s}Z_1, Z_2^p)$. Here s is chosen such that $qs \equiv 1 \pmod p$. Thus a matrix $\begin{pmatrix}-q&r\\p&s\end{pmatrix}$ of determinant -1 exists. We also have $(D^2 \times S^1)_2/(\mathbb{Z}/p) \overset{\cong}{\to} D^2 \times S^1$ by $[Z_2,Z_1] \mapsto (Z_1^{-q}Z_2, Z_1^p)$. Thus $L(p,q) = S^3/(\mathbb{Z}/p) \cong D^2 \times S^1 \cup_f D^2 \times S^1$ where $f: S^1 \times S^1 \to S^1 \times S^1$ and is given by $f(Z_2^{-s}Z_1, Z_2^p) = (Z_1^{-q}Z_2, Z_1^p)$. To put f in a more convenient form let $t_1 = Z_2^{-s}Z_1$, $t_2 = Z_2^p$. Then $Z_1^{-q}Z_2 = t_1^{-q}t_2^r$, $Z_1^p = t_1^p t_2^s$. Therefore $f: S^1 \times S^1 \to S^1 \times S^1$ is given by $f(t_1,t_2) = (t_1^{-q}t_2^r, t_1^p t_2^s)$.

<u>Remark.</u> We can choose M_i, L_i a meridian and longitude in $\partial(D^2 \times S^1)_i$ so $M_i = \{(t,1) \in \partial(D^2 \times S^1)_i\}$, $L_i = \{(1,t) \in \partial(D^2 \times S^1)_i\}$. Then under the pasting map $M_1 \mapsto \{(t^{-q},t^p) \in \partial(D^2 \times S^1)_2\}$, so $M_1 \sim -qM_2 + pL_2$. Thus this homology relation determines which lens space we are in.

We will apply this to find the Seifert invariants of the various G.S. fibered structures on $L(p,q)$.

<u>Theorem 4.4.</u> $L(p,q) \cong M(0;(\alpha_1,\beta_1),(\alpha_2,\beta_2))$ if

$$p = \det\begin{pmatrix}\alpha_1 & \alpha_2\\ -\beta_1 & \beta_2\end{pmatrix} = \alpha_1\beta_2 + \alpha_1\beta_2$$

$$q = \det\begin{pmatrix}\alpha_1 & \alpha_2'\\ -\beta_1 & \beta_2'\end{pmatrix} = \alpha_1\beta_2' + \beta_1\alpha_2'$$

where $\det\begin{pmatrix}\alpha_2 & \alpha_2'\\ \beta_2 & \beta_2'\end{pmatrix} = \alpha_2\beta_2' - \beta_2\alpha_2' = 1.$

Proof: $M(0;(\alpha_1,\beta_1),(\alpha_2,\beta_2)) \cong ((D^2 \times S^1) \cup (\text{annulus} \times S^1)) \cup (D^2 \times S^1)$
$= (D^2 \times S^1) \cup (D^2 \times S^1).$

We have homology relations $M_1 \sim \alpha_1 Q_1 + \beta_1 H_1 \quad M_2 \sim \alpha_2 Q_2 + \beta_2 H_2$
$L_1 \sim \alpha_1' Q_1 + \beta_1' H_1 \quad L_2 \sim \alpha_2' Q_2 + \beta_2' H_2$

where $\det\begin{pmatrix}\alpha_1 & \alpha_1'\\ \beta_1 & \beta_1'\end{pmatrix} = \det\begin{pmatrix}\alpha_2 & \alpha_2'\\ \beta_2 & \beta_2'\end{pmatrix} = 1.$ $Q_1 + Q_2 = \partial$ (section in annulus $\times S^1$) ~ 0 in homology, thus $Q_1 \sim -Q_2$. Also $H_1 \sim H_2$. Therefore

$$\begin{pmatrix}M_1\\ L_1\end{pmatrix} = \begin{pmatrix}\alpha_1 & \beta_1\\ \alpha_1' & \beta_1'\end{pmatrix}\begin{pmatrix}Q_1\\ H_1\end{pmatrix} = \begin{pmatrix}-\alpha_1 & \beta_1\\ -\alpha_1' & \beta_1'\end{pmatrix}\begin{pmatrix}Q_2\\ H_2\end{pmatrix}$$
$$= \begin{pmatrix}-\alpha_1 & \beta_1\\ -\alpha_1' & \beta_1'\end{pmatrix}\begin{pmatrix}\beta_2' & -\beta_2\\ -\alpha_2' & \alpha_2\end{pmatrix}\begin{pmatrix}M_2\\ L_2\end{pmatrix}$$
$$= \begin{pmatrix}-(\alpha_1\beta_2' + \beta_1\alpha_2') & \alpha_1\beta_2 + \beta_1\alpha_2\\ -(\alpha_1'\beta_2' + \beta_1'\alpha_2') & \alpha_1'\beta_2 + \beta_1'\alpha_2\end{pmatrix}\begin{pmatrix}M_2\\ L_2\end{pmatrix},$$

so by the remark preceding the theorem, Theorem 4.4 is proved.

Example: Let S^1 act on S^3 by $t(Z_1,Z_2) = (t^a Z_1, t^b Z_2)$ where $\gcd(a,b) = 1$. The isotropy subgroups are $\mathbb{Z}/a$ and $\mathbb{Z}/b$. By the above theorem, we get $e(S^3 \to S^3/S^1) \cong M(0; (a,a'),(b,b'))$ where $ab' + ba' = \pm 1$.

Exercise: The correct sign here is $ab' + ba' = +1$, so $e(S^3 \to S^3/S^1) = -(a'/a) - (b'/b) = -(1/ab)$.

5. Classification of Seifert Fiberable Manifolds.

With the previous section as preparation, we can now state the classification, up to (not necessarily fiber preserving) orientation preserving homeomorphism of Seifert and G.S. fiberable 3-manifolds.

Theorem 5.1.

1) $M(-1;(\alpha,\beta)) \cong M(0; (2,1),(2,-1),(-\beta,\alpha))$

$M(-2;(1,0)) \cong M(0;(2,1),(2,1);(2,-1),(2,-1))$.

2) The diffeomorphisms in 1) and Seifert fibered structures on lens spaces give the only examples of a 3-manifold having two non isomorphic Seifert fibrations.

3) $M(g;(0,1),(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)) \cong \#_{i=1}^{2g} (S^1 \times S^2) \# \#_{i=1}^{n} -L(\alpha_i,\beta_i)$ if $g \geq 0$

$\cong \#_{i=1}^{|g|} (S^1 \times S^2) \# \#_{i=1}^{n} -L(\alpha_i,\beta_i)$ if $g < 0$.

4) The only Seifert fibered manifold which is not connected sum prime is $M(-1;(1,0)) \cong \mathbb{R}P^3 \# \mathbb{R}P^3$.

Proof. For 1) and 3) see exercise 2. 2) and 4) will follow from our analysis of $\pi_1(M)$. See [O-V-Z], [O-R] In fact much more is true:

Theorem 5.2. (Waldhausen [Wa 1,2]) Let M_1,M_2 be Seifert fibered and not in the following list:

i) lens spaces

ii) $M(0;(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3))$

iii) $M(1;(1,0)) = T^3$

iv) As in part 1 of Theorem 5.1.

Then any homomorphism $M_1 \to M_2$ is isotopic to a fiber preserving homeomorphism.

6. The Fundamental Group of a Seifert Manifold.

Theorem 6.1. Let $M = M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$, then:

$$\pi_1(M) = \langle a_i,b_i,q_j,h \mid [h,a_i] = [h,b_i] = [h,q_j] = 1,$$
$$q_j^{\alpha_j}h^{\beta_j} = 1,\ q_1q_2,\ldots,q_m[a_1,b_1],\ldots,[a_n,b_n]h^{\bar{e}} = 1\rangle \quad \text{if} \quad g \geq 0$$

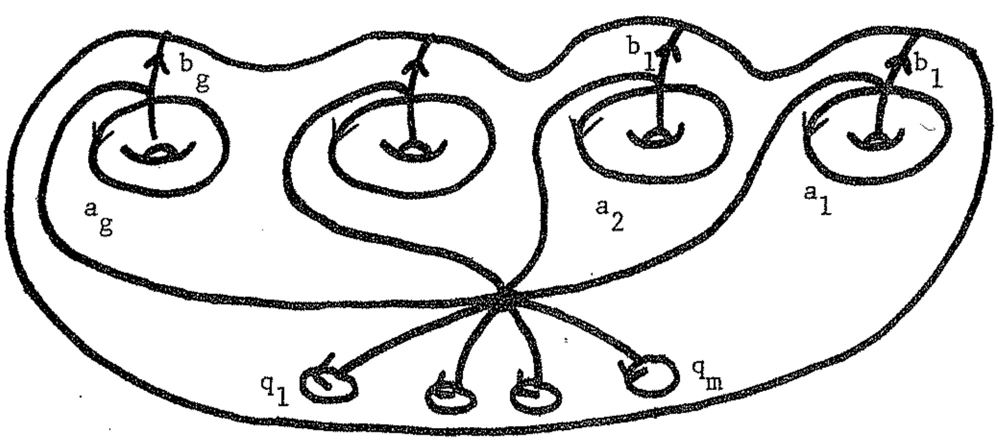
$$\pi_1(M) = \langle a_i,q_j,h \mid a_i^{-1}ha_i = h^{-1},\ [h,q_j] = 1,$$
$$q_j^{\alpha_j}h^{\beta_j} = 1,\ q_1,\ldots,q_m a_1^2,\ldots,a_{|g|}^2 = 1\rangle \quad \text{if} \quad g < 0,$$
$$(i=1,\ldots,|g|;\ j=1,\ldots,m).$$

Proof. We prove only the case $g \geq 0$. The proof for $g < 0$ is analogous. We shall apply Van Kampen's theorem to the representation

$$M = (F - mD^2) \times S^1 \cup T_1 \cup \ldots \cup T_m \qquad T_i = D^2 \times S^1.$$

$$\pi_1(F - mD^2) = \langle a_1,b_1,\ldots,a_g,b_g,q_1,\ldots,q_m \mid$$
$$q_1\ldots q_m[a_1,b_1]\ldots[a_g,b_g] = 1\rangle$$

where the a_i,b_i,q_j are represented in schematically in the figure below.



Then $\pi_1(F - mD^2 \times S^1) = \langle a_1,b_1,\ldots,a_g,b_g,q_1,\ldots,q_m,h \mid \prod_{j=1}^{m} q_j \prod_{i=1}^{g} [a_i,b_i] = 1,$

$$[h,a_i] = [h,b_j] = [h,q_j] = 1\rangle.$$

Claim. Pasting in T_j adds the relation $q_j^{\alpha_j} h^{\beta_j} = 1$.

Proof. By Van Kampen's theorem pasting in T_j adds a new generator t and two new relations

i) $q_j^{\alpha_j} h^{\beta_j} = 1$

ii) $q_j^{\alpha'_j} h^{\beta'_j} = t.$

The new generator t and relation ii) can be deleted by a Tietze transformation.

Corollary 6.2. If $g \geq 0$

$$H_1(M;\mathbb{Z}) = \langle A_i,B_i,Q_j,H \mid \alpha_j Q_j + \beta_j H = 0,\ Q_1 + \ldots + Q_m = 0\rangle$$

$$= \mathbb{Z}^{2g} \oplus \langle Q_j,H \mid \alpha_j Q_j + \beta_j H = 0,\ Q_1 + \ldots + Q_m = 0\rangle$$

$$= \mathbb{Z}^{2g} \oplus \operatorname{cok} \begin{pmatrix} 1 & 1 & \ldots & 1 & 0 \\ \alpha_1 & 0 & \ldots & 0 & \beta_1 \\ 0 & \alpha_2 & \ldots & 0 & \beta_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \ldots & \alpha_m & \beta_m \end{pmatrix}$$

In particular: If $e(M \to F) \neq 0$ then $H_1(M;\mathbb{Z}) = \mathbb{Z}^{2g} \oplus T$ with $|T| = \alpha_1,\ldots,\alpha_m \, |e(M \to F)|$.

If $e(M \to F) = 0$ and (M,F,π) is a Seifert fibration then

$H_1(M;\mathbb{Z})$ has free rank $2g+1$.

Proof: Cok A has order equal to $|\det A|$ if $\det A \neq 0$. Here $\det A = (-1)^m \alpha_1 \dots \alpha_m e(M \to F)$ by a simple induction. If $\det A = 0$ then rank $A = m$ if no α_i is 0.

Corollary 6.3. If $g \geq 0$ and (M,F,π) is a Seifert fibration, the following is a short exact sequence:

$$1 \to C \to \pi_1(M) \to \Gamma(g,\alpha_1,\dots,\alpha_m) \to 1$$

where C is the central cyclic subgroup of π generated by h and $\Gamma(g;\alpha_1,\dots,\alpha_n) = \langle a_1,b_1,\dots,a_g,b_g,q_1,\dots,q_j \mid q_j^{\alpha_j} = 1, \prod_j q_j \prod_i [a_i,b_i] = 1\rangle$.

Remark. $\Gamma(g;\alpha_1,\dots,\alpha_n)$ is a spherical, Euclidean, or hyperbolic crystallographic group according as $(2g-2) + \sum (\alpha_i - 1)/\alpha_i <, =, > 0$ respectively. We will see what this means and its significance for Seifert manifolds later.

We are now in a position to determine which Seifert manifolds are homology spheres. Assume $M = M(g;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ is a homology sphere. We can immediately conclude $e(M \to F) \neq 0$ and $g = 0$. For if $e(M \to F) = 0$ or $g > 0$, then by Corollary 6.2 $H_1(M;\mathbb{Z})$ is infinite. Also $g < 0$ cannot occur, for otherwise M would admit a connected 2-fold cover, implying $H^1(M;\mathbb{Z}/2) \neq \{0\}$. We have

$$1 = |H_1(M)| = \alpha_1, \ldots \alpha_n \mid \sum \beta_i/\alpha_i|$$

$$= |\beta_1\alpha_2 \ldots \alpha_n + \beta_2\alpha_1\hat{\alpha}_2 \ldots \alpha_n + \ldots + \beta_n\alpha_1 \ldots \alpha_{n-1}|.$$

By reversing orientation if necessary, we can assume $\sum_{i=1}^{n} \beta_i\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_n = 1$. Therefore the α_i's must be pairwise coprime, since if $d|\alpha_i$ and $d|\alpha_j$ then $d| \sum \beta_i\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_n$ and Hence $d|1$.

Moreover β_i is determined modulo α_i by

$$\beta_i\alpha_1 \ldots \hat{\alpha}_i \ldots \alpha_n \equiv 1 \bmod \alpha_i.$$

This completely determines the Seifert manifold since we know $e(M \to F)$ and $\beta_i \bmod \alpha_i$ for each i.

Conversely, if we have pairwise coprime α_i's we can find β_i satisfying $\sum \beta_i\alpha_i \ldots \hat{\alpha}_i \ldots \alpha_n = 1$. This proves:

<u>Theorem 6.4</u>. Given pairwise co-prime $\alpha_1, \ldots, \alpha_n$, there exists a unique Seifert manifold $M = \sum(\alpha_1, \ldots, \alpha_n)$ with the α_i's representing exceptional fibers and $e(\sum(\alpha_1, \ldots, \alpha_n) \to S^2) = -(1/(\alpha_1 \ldots \alpha_n)$ is a homology sphere.

<u>Example</u>. $\sum(2,3,5)$. Here $e(\sum(2,3,5) \to S^2) = -1/30$. Thus $\beta_1/2 + \beta_2/3 + \beta_3/5 = 1/30$ and we have

$$\sum(2,3,5) = M(0;(2,1),(3,1),(5,-4))$$

$$= M(0;(1,-1),(2,1),(3,1),(5,1)).$$

EXERCISES TO CHAPTER I

Exercise 1 (Homotopy classification of Lens spaces)

1) Show that a degree 1 map $L(p,q) \to L(p,q')$ is an orientation preserving homotopy equivalence.

2) Show that if $\phi: L(p,q) \to L(p,q')$ has degree d, then by suitable connected summing with the covering projection $S^3 \to L(p,q')$ you can get a map $\phi': L(p,q) \to L(p,q')$ of degree $d \pm p$. Thus if $\phi: L(p,q) \to L(p,q')$ exists of degree congruent to 1 mod p, then $L(p,q) \simeq L(p,q')$ (preserving orientation).

3) If $ab \equiv 1 \bmod p$ then the map $(Z_1,Z_2) \mapsto (Z_1^a,Z_2^b)/\|(Z_1^a,Z_2^b)\|$ of S^3 induces a map $L(p,q) \to L(p,b^2q)$ of degree ab, so by 1) and 2) $L(p,q) \simeq L(p,b^2q)$.

4) Conversely, show q is determined up to squares by the homotopy type of $L(p,q)$ as follows: Let $\beta: H^1(X;\mathbb{Q}/\mathbb{Z}) \to H^2(X:\mathbb{Z})$ be the connecting homomorphism for the coefficient sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. For $g_1,g_2 \in H^2(L(p,q);\mathbb{Z})$ define $\ell(g_1,g_2) = g_1 \cup \beta^{-1}(g_2) \in H^3(L(p,q);\mathbb{Z} \otimes \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$. Show $\ell(-,-)$ is well defined, and ℓ determines q up to squares as follows: for any generator $g \in H^2(L(p,q);\mathbb{Z})$ one has $\ell(g,g) = qx^2/p$ for some x prime to p.

Remark. Via Poincaré duality ℓ becomes the "torsion linking form," which is more generally defined for any closed oriented $2n+1$ manifold as

$$\ell: \begin{array}{ccc} \mathrm{Tor}\, H^{p+1}(M;\mathbb{Z}) \times \mathrm{Tor}\, H^{2n+1-p}(M;\mathbb{Z}) & & \\ \| \qquad\qquad\qquad \| & \to & \mathbb{Q}/\mathbb{Z}\,. \\ \mathrm{Tor}\, H_{2n-p}(M;\mathbb{Z}) \times \mathrm{Tor}\, H_p(M;\mathbb{Z}) & & \end{array}$$

The above is essentially the original approach, due to Rueff (Compositio Math. 6 (1938)), to classify $L(p,q)$'s up to homotopy.

*5) Use a similar approach to homotopy classify higher dimensional lens spaces $L^{2n+1}(p;q_1,\ldots,q_n)$, by replacing ℓ of 4) by an n-linear map $\ell(g_1,\ldots,g_n) = g_1 \cup \ldots \cup g_{n-1} \cup \beta^{-1}(g_n)$.

Exercise 2

1) Prove the following diffeomorphisms of G.S. fibered manifolds (preserving orientation but, of course, not preserving G.S. fibration), As stated in Section 5, these examples plus lens spaces give the only examples of non-equivalent G.S. fibrations of the same manifold.

a) $M(-1;(\alpha,\beta)) \cong M(0;(2,1),(2,-1),(-\beta,\alpha))$

b) $M(-2;(1,0)) \cong M(0;(2,1),(2,1),(2,-1),(2,-1))$

c) $M(g;(0,1),(\alpha_j,\beta_j),\ j = 1,\ldots,n) \cong \#_{i=1}^{k} S^1\times S^2 \# \#_{j=1}^{n} L(\alpha_j,-\beta_j)$
where $k = 2g$ $(g \geq 0)$ or $|g|$ $(g < 0)$.

Hints a) T^1Mb (the unit tangent bundle of the Möbius band) has two natural Seifert fibrations. The one is the projection $T^1Mb \to Mb$ and the other is given by the S^1-action on T^1Mb induced by an effective S^1-action on Mb. This gives two G.S. fibrations on any manifold of the form $T^1Mb \cup_{T^2} D^2 \times S^1$.

b) $T^1K\ell$ has two Seifert fibrations for the same reason T^1Mb does.

c) Define an operation of "connected sum along (0,1)-orbits" to show how to build up a G.S.-fibration on a connected sum of simpler non-Seifert G.S.-fibered manifolds. You know c) for $g = 0$ and $n = 1$ by classification of GS-fibrations on Seifert manifolds. Hence you know it also for $g = -1$, $n = 0$, by a). You only need it then for $g = 1$, $n = 0$. Find a suitable G.S. fibration on $((S^1 \times S^2) - 2D^2) \cup (S^2 \times I) \cong S^1 \times S^2 \# S^1 \times S^2$.

II. Further Examples

In this chapter we give two less basic examples of Seifert manifolds: Breiskorn complete intersections and the universal abelian cover of certain Seifert manifolds.

7. Breiskorn Complete Intersections.

Let $V(a_1,a_2,a_3) = \{(Z_1,Z_2,Z_3) \in C^3 \mid Z_1^{a_1} + Z_2^{a_2} + Z_3^{a_3} = 0\}$ where $a_1,a_2,a_3 \geq 2$. V has an isolated singular point at 0. We define the link of the singularity as $V(a_1,a_2,a_3) \cap S^5$ and denote it by $\sum(a_1,a_2,a_3)$. $\sum(a_1,a_2,a_3)$ has a natural S^1 action given by:

$$t(Z_1,Z_2,Z_3) = (t^{a/a_1}Z_1, t^{a/a_2}Z_2, t^{a/a_3}Z_3) \quad \text{where} \quad a = \text{lcm}(a_1,a_2,a_3).$$

This is an effective fixed point free action on $\sum(a_1,a_2,a_3)$. Therefore $\sum(a_1,a_2,a_3) \to \sum(a_1,a_2,a_3)/S^1$ is a Seifert fibration.

We can generalize this example by letting $A = (\alpha_{ij})^{i\,=\,1,\ldots,n-2}_{j\,=\,1,\ldots,n}$ be an $(n-2) \times n$ dimensional complex matrix and defining:

$$V_A(a_1,\ldots,a_n) = \{(Z_1,\ldots,Z_n) \in \mathbb{C}^n \mid \alpha_{i1}Z_1^{a_1} + \alpha_{i2}Z_2^{a_2} + \cdots + \alpha_{in}Z_n^{a_n} = 0,\ i = 1,\ldots,n-2\}.$$

<u>Proposition 7.1.</u> V_A is a 2-dimensional complex variety which is non-singular except at 0 iff each maximal $(n-2) \times (n-2)$ submatrix of A is non-singular.

Proof: Let $f_i(Z_1,\ldots,Z_n) = \sum_j \alpha_{ij} Z_j^{a_j}$ and $f = (f_1,\ldots,f_{n-2})$, $f\colon \mathbb{C}^n \to \mathbb{C}^{n-2}$ and $V_A = f^{-1}(0)$. We want to show: Df has rank $n-2$ at each point of $V_A - \{0\}$ iff each $(n-2) \times (n-2)$ submatrix of A is non-singular.

$$Df = (\partial f_i/\partial Z_j) = (\alpha_{ij} a_j Z_j^{a_j - 1}) = A \begin{pmatrix} a_1 Z_1^{a_1-1} & & 0 \\ & \ddots & \\ 0 & & a_n Z_n^{a_n-1} \end{pmatrix} .$$

Assume there is an $(n-2) \times (n-2)$ submatrix of Df that is singular. Without loss of generality, we can assume the first $n-2$ columns of A are linearly dependent. By a change of coordinates in $\mathbb{C}^{n-2}$, f is equivalent to $(\bar{f}_1,\bar{f}_2,\ldots,\bar{f}_{n-2})$ where $\bar{f}_1(Z_1,\ldots,Z_n) = \bar{\alpha}_{1n-1} Z_{n-1}^{a_{n-1}} + \bar{\alpha}_{1n} Z_n^{a_n}$. Then $\{\bar{f}_2 = \bar{f}_3 = \ldots = \bar{f}_{n-2} = Z_{n-1} = Z_{n-2} = 0\}$ is at least 1-dimensional (and in particular not zero) and contained in V_A. On this set Df has rank less than $n-2$, which is a contradiction.

Conversely, if every $(n-2) \times (n-2)$ submatrix is non-singular then Df has rank $n-2$ if at least $n-2$ Z_i's are non-zero. On $V_A - \{0\}$ if $Z_i = Z_j = 0$ then $Z_k^{a_k} = 0$ for the remaining $n-2$ indices k, which is a contradiction.

We call a matrix that satisfies the conditions of proposition 7.1 "good," and assume from now on A satisfies these conditions.

Definition $\sum_A(a_1,\ldots,a_n) = V_A \cap S^{2n-1}$. $\mathbb{C}^* = \mathbb{C} - \{0\}$ acts on $\mathbb{C}^n$ by $t(Z_1,\ldots,Z_n) = (t^{a/a_1}Z_1,\ldots,t^{a/a_n}Z_n)$ for all $t \in \mathbb{C}^*$ where $a = \mathrm{lcm}(a_i)$. This action preserves V_A. Therefore $S^1 \subseteq \mathbb{C}^*$ acts on $\sum_A(a_1,\ldots,a_n)$ making $\sum_A(a_1,\ldots,a_n) \to \sum_A(a_1,\ldots,a_n)/S^1$ a Seifert fibration.

Remark. $\sum_A(a_1,\ldots,a_n)$ is (as a Seifert manifold) independent of the choice of A with A good. This follows from the fact that $\{A | A \text{ is good}\}$ is connected. We denote $\Sigma_A(a_1,\ldots,a_n)$ by $\Sigma(a_1,\ldots,a_n)$. Of course the complex structure on V_A <u>does</u> depend on A.

<u>Theorem 7.2.</u> $\Sigma(a_1,\ldots,a_n) = M(g;s_1(t_1,\beta_1),\ldots,s_n(t_n,\beta_n))$ where

$$t_i = a/\operatorname{lcm}_{j\neq i}(a_j)$$

$$s_i = (\prod_{j\neq i} a_j)/\operatorname{lcm}_{j\neq i}(a_j)$$

$$g = (1/2)(2 + (n-2)(\Pi a_i)/a - \sum_{j=1}^{n} s_j)$$

$$e(\Sigma(a_1,\ldots,a_n) \to \Sigma(a_1,\ldots,a_n)/S^1) = -((\Pi a_i)/a^2).$$

Note that these four equations determine $\beta_i \pmod{t_i}$. The last equation can be written $\sum_j s_j\beta_j/t_j = \Pi(a_i)/a^2$. Dividing by the right side gives $\sum_j (a/a_j)\beta_j = 1$. Since $t_i = \gcd_{j\neq i}(a/a_j)$, t_i divides a/a_j if $i \neq j$, so $(a/a_i)\beta_i \equiv 1 \pmod{t_i}$.

<u>Example.</u> Assume the a_i are pairwise coprime. Then

$$t_i = a_i$$

$$s_i = 1$$

$$g = (1/2)(2 + (n-2)\cdot 1 - n) = 0$$

$$e = -1/(a_1\ldots a_n).$$

This is the Seifert homology sphere $\Sigma(a_1,\ldots,a_n)$. Thus our notation is consistent with that of the previous section.

Proof of Theorem 7.2. We require two facts from basic number theory which we state without proof.

i) if $\mathbb{Z}/m_i \subseteq S^1$, $i = 1,\ldots,k$ then $\bigcap_{i=1}^{k} \mathbb{Z}/m_i = \mathbb{Z}/\gcd(m_i)$

ii) $\gcd(m/m_i) = m/\mathrm{lcm}(m_i)$

$\mathrm{lcm}(m/m_i) = m/\gcd(m_i)$.

Note if $Z = (Z_1,\ldots,Z_n) \in V_A$ and $Z_i \neq 0$ for all i, then the isotropy subgroup $S^1_Z = \{1\}$. This follows from

$$S^1_Z = \mathbb{Z}/(a/a_1) \cap \mathbb{Z}/(a/a_2) \cap \cdots \cap \mathbb{Z}/(a/a_n)$$

$$= \mathbb{Z}/\gcd(a/a_i) = \mathbb{Z}/(a/\mathrm{lcm}(a_i)) = \{1\}.$$

Similarly, if $Z = (Z_1,\ldots,Z_n) \in V_A$ and $Z_i = 0$, $Z_j \neq 0$ for all $i \neq j$ then

$$S^1_Z = \mathbb{Z}/(a/a_1) \cap \cdots \cap \widehat{\mathbb{Z}/(a/a_i)} \cap \cdots \cap \mathbb{Z}/(a/a_n)$$

$$= \mathbb{Z}/\gcd_{j\neq i}(a/a_j) = \mathbb{Z}/(a/\mathrm{lcm}_{j\neq i}(a_j)) = \mathbb{Z}/t_i\ .$$

We also have, if $Z = (Z_1,\ldots,Z_n) \in V_A$ and $Z_i = Z_j = 0$ for $i \neq j$ then $Z = 0$, so $Z \notin \sum(a_1,\ldots,a_n)$.

We must next compute the number of orbits with isotropy $\mathbb{Z}/t_i$ or more precisely, the number s_i of orbits in $\sum(a_1,\ldots,a_n) \cap \{Z_i = 0\}$. We shall later show that (for fixed i) all these s_i orbits have the same β's.

Each orbit in $\sum(a_1,\ldots,a_n) \cap \{Z_1 = 0\}$ contains at least one point of the form $(0,r_2,Z_3,\ldots,Z_n)$ where $r_2 \in \mathbb{R}_+$.

a) In fact each orbit contains exactly $a/(a_2 t_1)$ such points since $\mathbb{Z}/(a/a_2)$ maps such a point to a similar point, while $\mathbb{Z}/\gcd_{i\neq 1}(a/a_i) = \mathbb{Z}/t_1$ maps such a point to itself.

b) $\sum(a_1,\ldots,a_n)$ contains exactly $a_3\ldots a_n$ such points since $\alpha_{i2}r_2^{a_2} + \alpha_{i3}Z_3^{a_3} + \ldots + \alpha_{in}Z_n^{a_n} = 0$, $i = 1,\ldots,n-2$, determines $(0, r_2^{a_2}, Z_3^{a_3},\ldots,Z_n^{a_n})$ up to a multiple. As we are on S^{2n-1}, r_2 is determined and Z_j is determined up to a_j-th roots of unity.

c) a) and b) imply there are $(a_3\ldots a_n)/(a/(a_2t_1)) = (a_2\ldots a_n t_1)/a = s_1$ orbits with $Z_1 = 0$.

To complete the proof we must verify the statements concerning g and e, and show the β's for all orbits in $\sum(a_1,\ldots,a_n) \cap \{Z_i = 0\}$ are the same for fixed i.

We have a map $\Phi: V_A(a_1,\ldots,a_n) - \{0\} \to V_A(1,\ldots,1) - \{0\}$ given by $(Z_1,\ldots,Z_n) \mapsto (Z_1^{a_1},\ldots,Z_n^{a_n})$. If this map induced a map $\varphi: \sum(a,\ldots,a_n) \to \sum(1,\ldots,1)$ we could use φ and theorem 3.3 to compute e. However $\Phi(S^{2n-1}) \not\subseteq S^{2n-1}$ and we must show Φ can be nevertheless used to induce such a φ. We have $\mathbb{R}_+ \subseteq \mathbb{C}^*$ and

$$\begin{array}{ccc} \sum_A(a_1,\ldots,a_n) & \hookrightarrow & V_A(a_1,\ldots,a_n) - \{0\} \\ & \searrow^{\cong} & \downarrow \\ & & (V_A(a_1,\ldots,a_n) - \{0\})/\mathbb{R}_+ . \end{array}$$

Denote $\sum_A(a_1,\ldots,a_n)/S^1 \cong (V_A(a_1,\ldots,a_n) - \{0\})/(\mathbb{R}_+ \times S^1)$

$$\cong (V_A(a_1,\ldots,a_n) - \{0\})/\mathbb{C}^*$$

by $P_A(a_1,\ldots,a_n)$.

We have:

$$\begin{array}{ccc} V_A(a_1,\ldots,a_n) - \{0\} & \xrightarrow{\Phi} & V_A(1,\ldots,1) - \{0\} \\ \downarrow /\mathbb{R}_+ & & \downarrow /\mathbb{R}_+ \\ \sum_A(a_1,\ldots,a_n) & \xrightarrow{\varphi} & \sum_A(1,\ldots,1) \\ \downarrow /S^1 & & \downarrow /S^1 \\ P_A(a_1,\ldots,a_n) & \xrightarrow{\bar{\varphi}} & P_A(1,\ldots,1) \end{array}$$

φ is S^1-equivariant if we let S^1 act non effectively on $\sum(1,\ldots,1)$ by $t(Z_1,\ldots,Z_n)=(t^aZ_1,\ldots,t^aZ_n)$, $a = \mathrm{lcm}(a_i)$. Thus the degree of $\varphi|$(non-singular fiber) is a. The degree of φ = degree $\Phi = a_1\ldots a_n$. Therefore

$$e(\textstyle\sum(a_1,\ldots,a_n) \to P(a_1,\ldots,a_n)) = ((a_1\ldots a_n)/a^2)e(\sum(1,\ldots,1) \to P(1,\ldots,1))$$

$$= -(a_1\ldots a_n)/a^2$$

by Theorem 3.3 and since $\sum(1,\ldots,1) \to P(1,\ldots,1)$ is the Hopf fibration.

$P(a_1,\ldots,a_n) \to P(1,\ldots,1)$ is a $(\Pi a_i)/a$-fold branched covering. The branching occurs over $Z_i = 0$ over which we have s_i points in $P(a_1,\ldots,a_n)$ (because points in $P(a_1,\ldots,a_n)$ are, by definition, the same thing as orbits in $\sum(a_1,\ldots,a_n)$). The standard "Hurewitz formula" for the Euler characteristic of a branched cover thus gives

$$\chi(P(a_1,\ldots,a_n)) = ((\Pi a_i)/a)(\chi(P(1,\ldots,1)) - n) + \textstyle\sum s_i$$

$$= ((\Pi a_i)/a)(2-n) + \textstyle\sum s_i \ .$$

Since $\chi = 2 - 2g$, this gives the claimed value of g.

The fibers of Φ are the orbits of the natural $H = (\mathbb{Z}/a_1 \times \ldots \times \mathbb{Z}/a_n)$-action on $V_A - \{0\}$. (H acts by multiplication by a_i-th roots of unity in the i-th coordinate). This action on $V_A - \{0\}$ induces actions of H also on $\sum_A(a_1,\ldots,a_n)$ and $P_A(a_1,\ldots,a_n)$, and the fibers of φ and $\bar{\varphi}$ are therefore also the orbits of this action. In particular, for fixed i, $0 \leq i \leq n$, the orbits in $\sum_A(a_1,\ldots,a_n)$ with $Z_i = 0$ correspond to points in $P_A(a_1,\ldots,a_n)$ with $Z_i = 0$ which are all related by this H-action, since $P_A(1,\ldots,1)$ has exactly one point with $Z_i = 0$. Thus the H-action permutes the (t_i,β_i)-orbits of $\sum_A(a_1,\ldots,a_n)$ transitively, so the β's are the same for these orbits.

The whole proof can be expressed a bit more concisely in terms of this H-action, see [N-R], but the elementary nature of the computation is then even more obscured than by the presentation given here.

8. Universal Abelian Covers.

Lemma 8.1. If $M' \overset{p}{\to} M$ is a finite covering of a Seifert fibered manifold M, then M' is Seifert fibered by the components of p^{-1} of fibers of M.

Proof. Any finite coverings of one of our "standard models" (Seifert fibering of $D^2 \times S^1$) will be a disjoint union of solid tori $D^2 \times S^1$, each with some standard Seifert fibering induced on it.

As an example of Lemma 8.1, assume $M = M(0;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ with $e(M \to F) \neq 0$. Let $\bar{M} \overset{p}{\to} M$ be the universal abelian cover of M. Therefore the covering transformation group is $H_1(M)$, which has order $\alpha_1 \dots \alpha_n \, |e(M \to F)|$. By the lemma $\bar{M}$ has an induced Seifert fibering $\bar{M} \to \bar{F}$.

This example and the Breiskorn complete intersections are two examples of Seifert fibrations arising in a "natural" way. As the next theorem shows, even though these two examples arise from very different situations, surprisingly they are the same.

Theorem 8.2. Let $M = M(0;(\alpha_1,\beta_1),\dots,(\alpha_n,\beta_n))$ with $\hat{e} = e(M \to F) \neq 0$. Reverse the orientation of M if necessary to make $e < 0$. Let $\bar{M} \overset{p}{\to} M$ be the universal abelian cover. Then $\bar{M} = \sum(\alpha_1,\dots,\alpha_n)$

Proof: We must recall some facts from covering space theory. Let X be a "nice" space (i.e. X has a universal cover) and assume X is connected.

The normal coverings of X with covering transformation group F are classified by homomorphisms $\pi_1(X) \to F$. If the covering is $Y \to X$, then the components of Y are in 1-1 correspondence with the cosets of $\text{Im}\ \pi_1(x) \subseteq F$. Namely, given $Y \to X$, the exact homotopy sequence gives:

$$\pi_1(\text{fiber}) = \{1\} \to \pi_1(Y) \to \pi_1(X) \to \pi_0(\text{fiber}) = F \to \pi_0(Y) \to \{1\}.$$

Given $\pi_1(X) \to F$, one can construct Y as $\tilde{X} \times_{\pi_1(X)} F$ ($\tilde{X}$ = universal cover of X). Then $Y \to \tilde{X}/\pi_1(X) = X$. The universal abelian cover is classified by the homomorphism $\pi_1(X) \to H_1(X)$.

Returning to the situation of the theorem, we have a commutative diagram

$$\begin{array}{ccc} \bar{M} & \xrightarrow{p} & M \\ {\scriptstyle \bar{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \bar{F} & \longrightarrow & F \end{array}$$

We want to compute the fiber degree f and the base degree b to apply Theorem 3.3. Recall

$$H_1(M) = \langle Q_1,\ldots,Q_n,H \mid \alpha_i Q_i + \beta_i H = 0,\ Q_1 + \ldots + Q_n = 0\rangle$$

has order $\alpha_1 \cdots \alpha_n |e|$ so the total degree bf is $\alpha_1 \cdots \alpha_n |e|$. Let O be a non-exceptional fiber. The induced coverings $p^{-1}(O) \to O$ is classified by $\pi_1(O) \to H_1(M)$. Thus to compute f and b we must compute $|\text{Im}\ \pi_1(O)|$ and $|H_1(M)\colon \text{Im}\ \pi_1(O)|$.

As we have already seen $|\mathrm{Im}\ \pi_1(0)| = |\langle H\rangle|$. Thus

$$|H_1(M)\colon \mathrm{Im}\ \pi_1(0)| = |H_1(M)/ \langle H\rangle|$$

$$= |\langle Q_1,\ldots,Q_n |\ \alpha_i Q_i = 0,\ Q_1 + \ldots + Q_n = 0\rangle|$$

$$= |(\mathbb{Z}/\alpha_1 \times \ldots \times \mathbb{Z}/\alpha_n)/\langle Q_1 + \ldots + Q_n\rangle|$$

$$= (\Pi\alpha_i)/\ \mathrm{lcm}\ \alpha_i$$

$$= (\Pi\alpha_i)/a\ ,\ \text{where}\ \ a = \mathrm{lcm}(\alpha_i)$$

$$|H| = (\Pi\alpha_i))|e|/(\Pi\alpha_i/a) = a|e|.$$

Thus we have $f = a|e|$ and $b = \Pi\,\alpha_i/a$.

Using Theorem 3.3, we can calculate

$$e(\bar{M} \to \bar{F}) = b/f\ e(M \to F)$$

$$= (((\Pi\alpha_i)/a)/a|e|)e(M \to F)$$

$$= -(\Pi\alpha_i)/a^2.$$

Now let O_i be the i-th exceptional fiber. If $\det\begin{pmatrix}\alpha_i & \beta_i\\ \alpha_i' & \beta_i'\end{pmatrix} = 1$, then the homology class of O_i is

$$O_i = \alpha_i' Q_i + \beta_i' H\ .$$

Hence $|H_1(M)\colon \text{Im}\ \pi_1(O_i)| = |H_1(M)/ <O_i>|$

$=|<Q_1,\ldots,Q_n,H|\ \alpha_j Q_j + \beta_j H = 0,\ j = 1,\ldots,n,\ Q_1 + \cdots + Q_n = 0,\ \alpha_i' Q_i + \beta_i' H = 0>|$

$= |<Q_1,\ldots,\hat{Q}_i,\ldots,Q_n|\ \ \alpha_j Q_j = 0,\ j = 1,\ldots,\hat{i},\ldots,n,\ Q_1 + \cdots + \hat{Q}_i + \cdots + Q_n = 0>|$

$= (\prod_{j\neq i} \alpha_j)/ \text{lcm}_{j\neq i}\, \alpha_j = s_i$. Thus $p^{-1}(O_i)$ consists of s_i fibers, each of which covers O_i with degree

$$\begin{aligned}(\Pi\alpha_j)|e|/s_i &= (\Pi\alpha_j)|e|/(\prod_{j\neq i} \alpha_j)/(\text{lcm}_{j\neq i}\, \alpha_j)\\ &= \alpha_i |e|\ \text{lcm}_{j\neq i}\, \alpha_j\\ &= (\alpha_i/t_i)a|e| .\end{aligned}$$

Let $\bar{O}_i$ be one of the s_i fibers in $\bar{M}$ which cover O_i and U be a neighborhood of O_i and $\bar{U}$ a neighborhood of $\bar{O}_i$. Let H and $\bar{H}$ denote non-exceptional fibers in M and $\bar{M}$. $\bar{O}_i$ is a $(\alpha_i/t_i)a|e|$-fold cover of O_i and $\bar{H}$ is an $a|e|$-fold cover of H. In $H_1(U)$, $\alpha_i O_i \sim H$. $\rho_*\colon H_1(\bar{U}) \to H_1(U)$ is injective, hence $\alpha_i a|e|\bar{O}_i \sim (\alpha_i/t_i)a|e|\bar{H}$ in $H_1(\bar{U})$. Therefore in $H_1(\bar{U})$, $t_i\bar{O}_i \sim \bar{H}$, and $\bar{O}_i$ is a (t_i,β_i) fiber for some β_i. (Note that the β's for all s_i of these fibers are equal since these fibers are transitively permuted by the covering transformations.)

$\bar{F}$ is a $(\Pi\alpha_i)/a$-fold branched cover of F. The branching is at n points of F over which we have respectively $s_1,\ldots,s_n$ points in $\bar{F}$. Then

$$X(\bar{F}) = ((\Pi\alpha_i)/a)(X(F) - n) + \sum s_j .$$

As $X(F) = 2$, this gives the desired g, and as in the proof of Theorem 7.2, the computed data completely determines $\bar{M}$.

III. Crystallographic Groups

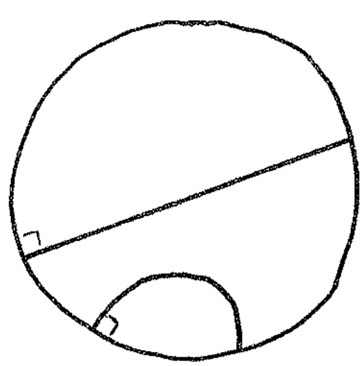
In this chapter we define and characterize 2-dimensional crystallographic groups. In the second section we use this to show that $\pi_1(M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ determines the Seifert invariant of $M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$. We begin this chapter with a short discussion of models for the three basic 2-dimensional geometries, Spherical, Euclidean and Hyperbolic.

1) Spherical (S^2): The sphere S^2 is a conformal model for this geometry. For geodesics we take great circles. If so desired, to avoid having two lines intersect in more than one point, this geometry can be projected onto $\mathbb{R}P^2$.

2) Euclidean ($\mathbb{E}^2$): This is the usual geometry on $\mathbb{R}^2$.

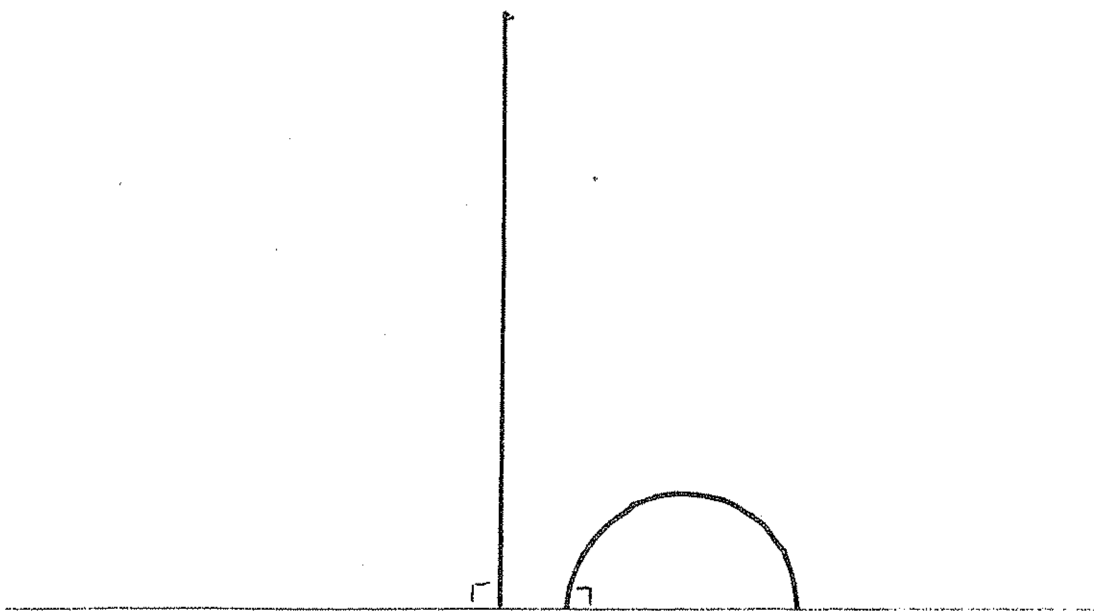
3) Hyperbolic ($\mathbb{H}^2$): There are two commonly used conformal models for $\mathbb{H}^2$: the Poincaré disc model and the upper half-plane model.

   a) Poincaré disc model: Here the underlying space is the interior of the unit disc D^2. Geodesics are either circular arcs that intersect the boundary of D^2 at right angles or diameters. Hyperbolic length is related to Euclidean length by $ds^2 = ds^2_{Eucl}/(1-r^2)$ where ds^2_{Eucl} is the Euclidean metric.



Poincaré disc model of $\mathbb{H}^2$ with examples of geodesics.

b) Upper-half plane model: The underlying space of this model is the upper-half plane in $\mathbb{R}^2$ i.e. $\{(x,y) \varepsilon \mathbb{R}^2 \,|\, x > 0\}$. Geodesics are either circular arcs that intersect the x-axis at right angles or vertical lines. In this model Hyperbolic distance is related to Euclidean distance by $ds^2 = ds^2_{Eucl}/y$.



Upper-half plane model of $\mathbb{H}^2$
with examples of geodesics.

Note: $\mathrm{Isom}^+(\mathbb{H}^2)$ = {conformal orientation preserving homeomorphisms of $\mathbb{H}^2$}

= {Moebius transformations}

$\cong PSL(2,\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \,|\, a,b,c,d \;\varepsilon\; \mathbb{R},\; ad-bc = \{1\}/\{\pm I\}.$

Here $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \varepsilon\; PSL(2,\mathbb{R})$ acts in the upper-half plane model by the Moebius transformation $z \mapsto (az+b)/(cz+d)$.

9. Crystallographic Groups

Definition 9.1. A 2-dimensional (Spherical, Euclidean or Hyperbolic) crystallographic group is a discrete subgroup $\Gamma \subset \mathrm{Isom}^+(X)$ $(X = S^2, \mathbb{E}^2, \mathbb{H}^2)$ such that Γ acts properly discontinuously on X and X/Γ is compact. More precisely this is an orientation preserving crystallographic group, but we drop the extra adjectives for brevity.

Theorem 9.2. As an abstract group, a crystallographic group Γ is either finite cyclic (if $X = S^2$) or is isomorphic to a unique group of the form

$$\Gamma(g,\alpha_1,\ldots,\alpha_n) = \langle a_1,b_1,\ldots,a_g,b_g,q_1,\ldots,q_n \mid q_j^{\alpha_j} = 1 \ j=1,\ldots,n,$$

$$q_1\cdots q_n \prod_{i=1}^{g} [a_i,b_i] = 1\rangle$$

with $g \geq 0$ and $\alpha_i \geq 2$. We assume that if $g = 0$ then $n \geq 3$. Moreover this is a spherical, Euclidean or hyperbolic group according as $\chi = 2 - 2g - \sum_{i=1}^{n} \frac{\alpha_i - 1}{\alpha_i}$ satisfies $\chi > 0$, $\chi = 0$, $\chi < 0$. Furthermore all such groups $\Gamma(g;\alpha_1,\ldots,\alpha_n)$ occur as crystallographic groups. The above condition on χ gives the following possibilities:

Spherical case $(\chi > 0)$. We must have $g = 0$, $n = 3$ and $\sum_{i=1}^{n} \frac{1}{\alpha_i} > 1$. This gives possibilities

$$(g;\alpha_1,\alpha_2,\alpha_3) = (0;2,2,n)$$

$$= (0;2,3,3)$$

$= (0;2,3,4)$

$= (0;2,3,5)$

Euclidean Case ($\chi = 0$). In this case the only possibilities are

$$(g;\alpha_1.,,,.\alpha_n) = (1;\ )$$
$$(0;2,4,4)$$
$$(0;2,3,6)$$
$$(0;3,3,3)$$
$$(0;2,2,2,2).$$

Hyperbolic Case ($\chi < 0$). This case consists of all possibilities not previously listed.

Notation: If $g = 0$ we abbreviate $\Gamma(0;\alpha_1,\ldots,\alpha_n)$ to $\Gamma(\alpha_1,\ldots,\alpha_n)$.

Example. We can easily realize the spherical and Euclidean cases as orientation preserving isometries of S^2 and $\mathbb{E}^2$. The spherical crystallographic groups can be realized as the regular polyhedral groups, i.e. isometries of regular spherical polyhedra. More precisely we have the following chart:

Regular Polyhedron	Isometry Group	Order of Group
n-gonal dihedron*	$\Gamma(2,2,n) \cong D_{2n}$	2n
tetrahedron	$\Gamma(2,3,3) \cong A_4$	12
cube	$\Gamma(2,3,4) \cong S_4$	24
dodecahedron	$\Gamma(2,3,5) \cong A_5$	60

$\Gamma(2,3,4)$ can also be realized as the isometry group of the octahedron and $\Gamma(2,3,5)$ can also be realized as the isometry group of the icosohedron.

In the Euclidean case let $\overline{\Gamma}(\alpha_1,\alpha_2,\alpha_3)$ be the group generated by reflections in the sides of a triangle with angles π/α_1, π/α_2, π/α_3, e.g. $\overline{\Gamma}(2,4,4)$ is generated by reflections in the sides of a triangle with angles $\pi/2$, $\pi/4$, $\pi/4$. Then $\Gamma(\alpha_1,\alpha_2,\alpha_3) = (\overline{\Gamma}(\alpha_1,\alpha_2,\alpha_3))^+$ i.e. $\Gamma(\alpha_1,\alpha_2,\alpha_3)$ is the orientation preserving subgroup of $\overline{\Gamma}(\alpha_1,\alpha_2,\alpha_3)$. $\overline{\Gamma}(2,2,2,2)$ is the group generated by reflections in the sides of a rectangle. Then $\Gamma(2,2,2,2) = (\overline{\Gamma}(2,2,2,2))^+$.

Exercise. Show that the spherical and Euclidean groups described above have the abstract description claimed in Theorem 9.2.

Remark. The above construction of the Euclidean groups extends to give $\Gamma(\alpha_1,\ldots,\alpha_n)$ as the orientation preserving subgroup of the group $\overline{\Gamma}(\alpha_1,\ldots,\alpha_n)$ generated by reflections in the sides of a spherical, Euclidean or hyperbolic n-gon with angles $\pi/\alpha_1,\ldots,\pi/\alpha_n$.

* This is the regular spherical polyhedron with two regular n-gonal faces. Each face is a hemisphere and the vertices are regularly spaces around the equator.

<u>Proof of Theorem 9.2</u>. Throughout this proof X is one of S^2, $\mathbb{E}^2$, $\mathbb{H}^2$. Let Γ be any discrete group acting properly discontinuously on X with X/Γ a compact surface. Here by properly discontinuous we mean for all $x,y \in X$ one can find neighborhoods N_x, N_y of x and y such that $N_x \cap gN_y = \phi$ for all but finitely many $g \in G$. $X \overset{\pi}{\to} X/\Gamma$ is a branched covering. The branching occurs over points on which Γ acts with non-trivial isotropy. Call the branch points $x_1,\ldots,x_n \in X/\Gamma$. Then $\pi|_{\pi^{-1}(X/\Gamma-\{x_1,\ldots,x_n\})}$ is a genuine covering map. Since a finite group of orientation preserving homeomorphism acting on $\mathbb{R}^2$ must be a rotation (see [ker]) π restricted to each component of $\pi^{-1}(D_i)$ (D_i is a small disc about x_i) "looks like" $D^2 \to D^2$ given in complex coordinates by $z \mapsto z^{\alpha_i}$.

From X/Γ cut out the discs D_i and in X cut out $\pi^{-1}(D_i)$. What remains is a genuine cover. We want to replace D_i and $\pi^{-1}(D_i)$ by complexes in such a way that this new space is a cover with covering transformation group Γ. We do this by replacing D_i by an Eilenberg Maclane complex $K_i = K(\mathbb{Z}/\alpha_i,1)$ for each i, pasting along a generator $S^1 \to K_i$ and correspondingly replacing $\pi^{-1}(D_i)$ by the universal cover $\tilde{K}_i$ of K_i pasting along the cover of S^1 in K_i. Call the result of this pasting $Y \to Y'$. Clearly Y is a cover of Y' and, since in X we replaced contractible spaces by contractible spaces, $X \simeq Y$. Thus Y is the universal cover of Y'. Γ acts properly discontinuously and freely on Y and by looking at this action on a fiber not lying above a point of D_i we see Γ is the group of covering transformations of Y. Thus $Y' \cong Y/\Gamma$ and $\Gamma \cong \pi_1(Y/\Gamma)$. By Van Kampen's theorem

$$\pi_1(Y/\Gamma) = \langle a_1,b_1,\ldots,a_g,b_g,q_1,\ldots,q_n \mid q_j^{\alpha_j} = 1, j = 1,\ldots,n,$$

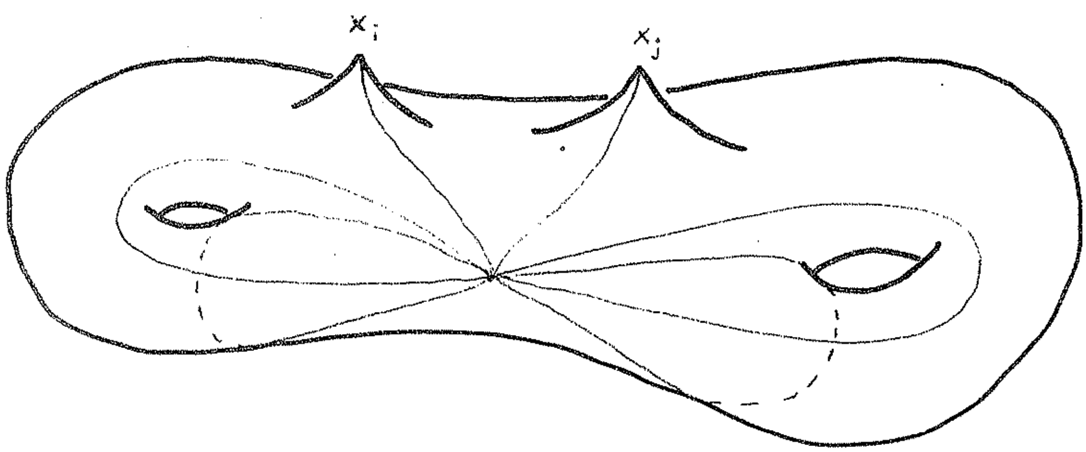
$$q_1 \cdots q_n \prod_{i=1}^{g} [a_i,b_i] = 1\rangle$$

and hence Γ has the desired presentation.

Remark: 1) In the Euclidean and hyperbolic cases Y is contractible so Y/Γ is a $K(\Gamma,1)$.

2) Notice we have proved a slightly stronger statement in that we did not assume $\Gamma \subset \mathrm{Isom}^+(X)$.

Now we assume $\Gamma \subset \mathrm{Isom}^+(X)$. X/Γ is as shown where x_i and x_j are branch points.



If we cut X/Γ along the geodesic paths shown, we get a polygon with $4g + 2n$ sides which is a fundamental domain. The sum of the angles of the polygon is $2\pi + \sum_{i=1}^{n} 2\pi/\alpha_i$. A special case of the Gauss-Bonnet formula states: if P is an m-sided polygon with angles $\theta_1,\ldots,\theta_m$, then $K(\text{area of } P) = \pi(2-m) + \sum_{i=1}^{m} \theta_i$ where $K = -1, 0, 1$ accordingly as $X = \mathbb{H}^2$, $\mathbb{E}^2$, S^2. In our case

$$K(\text{area of } P) = (2-(4g+2n))\pi + 2\pi(1 + \sum_{i=1}^{n} 1/\alpha_i)$$
$$= 2\pi(2-2g - \sum_{i=1}^{n} (\alpha_i-1)/\alpha_i).$$

Therefore X classifies what space Γ acts on in the manner stated in the theorem.

To finish the proof of Theorem 9.2 we must show:

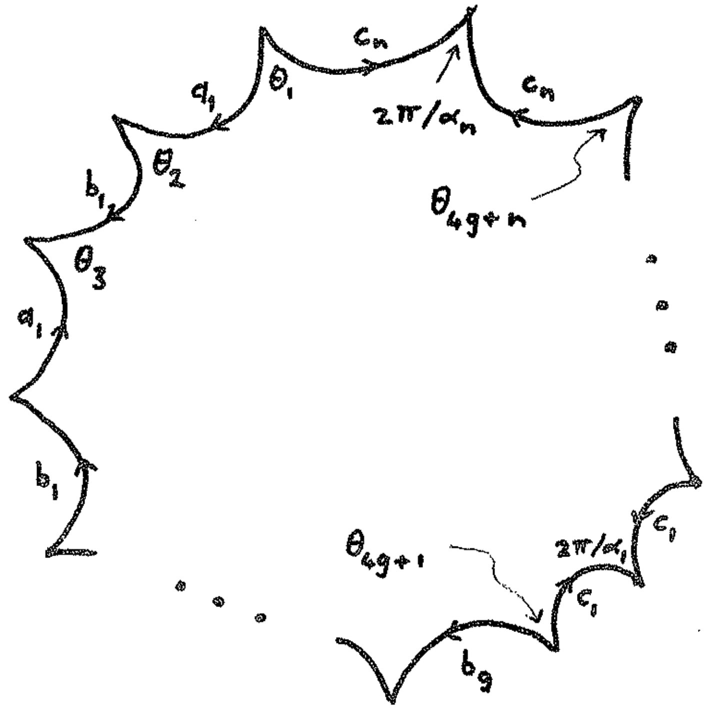
a) Given $\Gamma = \Gamma(g;\alpha_1,\ldots,\alpha_n)$ then Γ can be embedded in $\text{Isom}^+(X)$ as a crystallographic group;

b) The abstract group $\Gamma(g;\alpha_1,\ldots,\alpha_n)$ uniquely determines $g,\alpha_1,\ldots,\alpha_n$

We have already shown by example that if Γ is spherical c Euclidean then $\Gamma \subseteq \text{Isom}^+(X)$. Thus assume $X = \mathbb{H}^2$ and $2-2g - \sum_{i=1}^{n} (\alpha_i-1)/\alpha_i < 0$. To show $\Gamma \subseteq \text{Isom}^+(\mathbb{H}^2)$ it suffices to construct a polygon P in $\mathbb{H}^2$ such that P is a fundamental domain for X/Γ. Clearly P must be a polygon of the form:

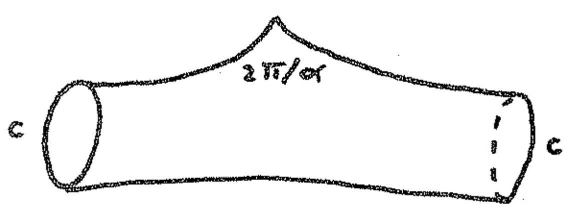
$\sum_{1}^{4g+n} \theta_i = 2\pi$;

similarly labelled sides are of equal length.



We leave the proof of the existence of such a polygon to the reader. However a reference for such a proof is [Gr].

An easier way, due to Scott Wolpert, to show $\Gamma \subseteq \mathrm{Isom}^+(\mathbb{H}^2)$ is to construct the orbit space directly. This can be done for $g > 0$ by glueing together pieces of the form



c is the length of each S^1 boundary component. This length is fixed but arbitrary. Again the details of this construction are left to the reader.

Finally we must prove b). Note in the Spherical case we have

$\Gamma(2,2,n) = D_{2n}$ of order $2n$

$\Gamma(2,3,3) = A_4$ of order 12

$\Gamma(2,3,4) = S_4$ of order 24

$\Gamma(2,3,5) = A_5$ of order 60.

It is a simple exercise to show no two of these groups are isomorphic. Hence this proves b) in the Spherical case.

Now assume $2 - 2g - \sum_{i=2}^{n} (\alpha_i - 1)/\alpha_i \leq 0$ i.e. $\Gamma \subsetneq \mathrm{Isom}^+(X)$ where $X = \mathbb{R}^2$ or $\mathbb{H}^2$. We claim that if $\{1\} \neq F \subseteq \Gamma$ is any finite (cyclic) subgroup then F is conjugate to a subgroup of a unique one of the $\langle q_i \rangle$. To see this note if F is finite then F has a unique fixed point in X. (For $X = \mathbb{R}^2$ see [Ker]; for $X = \mathbb{H}^2$ see [Hel;p. 75]). This fixed point is a branched point for $X \xrightarrow{\pi} X/\Gamma$ so it lies over one of $x_1,\ldots,x_n$. Say $\pi(y) = x_1$. q_1 is a rotation about some point in $\pi^{-1}(x_1)$ i.e. about some point γy. Then $\gamma F \gamma^{-1}$ fixes y and hence is in $\Gamma_y = \langle q_1 \rangle$ which proves the claim. In particular any maximal finite subgroup is conjugate to a unique one of the $\langle q_i \rangle$. Thus if we consider the set of conjugacy classes of all maximal finite subgroups of Γ, the number of classes is n and the orders of a representative from each conjugacy class will give us the α_i's. Finally if N is the normal subgroup generated by all elements of finite order then Γ/N has presentation:

$$\Gamma/N = \langle a_1,b_1,\ldots,a_g,b_g \mid \prod_{i=1}^{g} [a_i,b_i] = 1 \rangle,$$

which allows us to determine g.

Theorem 9.3. Let $\Gamma = \Gamma(g;\alpha_1,\ldots,\alpha_n)$ be a crystallographic group (if $g = 0$ we assume $n \geq 3$) and let X be the corresponding geometry, i.e. X is either S^2, $\mathbb{E}^2$ or $\mathbb{H}^2$. Thus $\Gamma \subsetneq \text{Isom}^+(X) = G$. Then under the above hypotheses we have:

$$G/\Gamma = M(g;(1,2g-2), (\alpha_1,\alpha_1-1),\ldots,(\alpha_n,\alpha_n-1)).$$

Note: $$e(G/\Gamma \to F) = 2 - 2g - \sum \frac{\alpha_i-1}{\alpha_i}$$
$$= \pm \frac{1}{\pi} \text{ vol}(X/\Gamma) \quad (\text{if } X \neq \mathbb{E}^2).$$

This number is called $\chi(\Gamma)$ and was first defined by C.T.C. Wall, see [W].

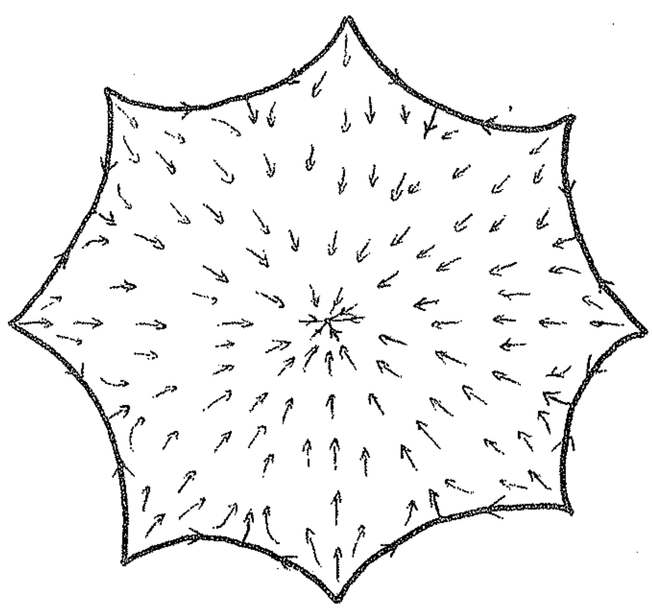
Proof G acts by isometries on X and hence on T^1X the unit tangent bundle of X. This action on T^1X is simply transitive, i.e. given $v_1, v_2 \in T^1X$ there exists a unique $g \in G$ such that $gv_1 = v_2$. Thus if we fix $v_o \in T^1X$ the map $\Phi : G \to T^1X$ given by $\Phi(g) = g \cdot v_o$ is an isomorphism. Here $g \cdot v_o$ is the action of G on T^1X induced by the action of $\Gamma \subset \text{Isom}^+(X)$ on X. If Γ acts on G by left multiplication and on T^1X as just described Φ is Γ-equivariant. Therefore $G/\Gamma \cong T^1X/\Gamma$ and we can describe the Seifert fibered structure as the natural projection $T^1X/\Gamma \overset{\pi}{\to} X/\Gamma$.

If $x \in X/\Gamma$ is the image of $y \in X$, then the fiber over x of the Siefert fibration is:

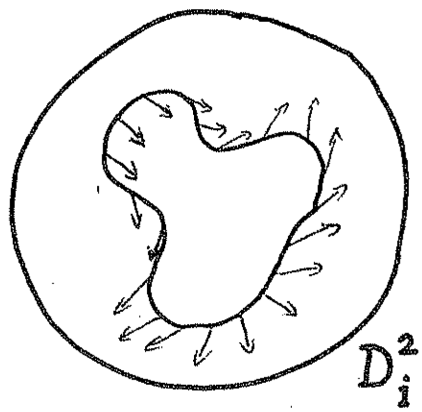
$$\pi^{-1}(x) = T^1_y(X)/\Gamma_y = \begin{cases} T^1_yX/(\mathbb{Z}/\alpha_i) & \text{if } x = x_i \text{ is singular} \\ T^1_yX & \text{otherwise.} \end{cases}$$

Thus $T^1X/\Gamma = M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ for some $\beta_1,\ldots,\beta_n$. Hence we need only determine the β_i. (In fact, our determination of the β_i will also give a second proof that the α_i are correct.)

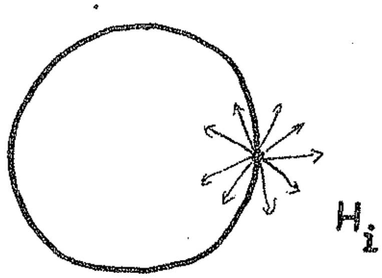
Before continuing the proof, we look at an example. Let Γ be the fundamental group of an orientable surface of genus 2, so $X/\Gamma = F_2$, an orientable surface of genus 2. The Seifert fibration is $T^1F_2 \xrightarrow{\pi} F_2$. It was classically known that $T^1F_2 = M(2;(1,2))$, but let us see how we would prove this from "first principles." To determine the Seifert invariant we must choose a section in $T^1F_2 - \{\bigcup_{i=1}^{n} T_i\}$ where the T_i are disjoint neighborhoods of suitable fibers i.e. disjoint solid tori. This is equivalent to choosing a unit tangent vector field on $F_2 - \{\bigcup_{i=1}^{n} D_i^2\}$ where as the disks D_i^2 we take small disks about the critical points of the vector field. Such a vector field drawn on a fundamental domain looks as follows:



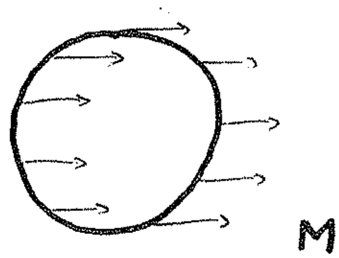
After identifying edges there are six critical points. Each critical point represents a deleted fiber. A solid torus neighborhood T_i of such a fiber is $T_i = T^1D_i^2$, so points of T_i are unit tangent vectors to the disk D_i. Thus the following picture represents a typical closed curve in $T_i = T^1D_i^2$.



Recall in ∂T_i we have the homology relation $\alpha_i Q_i + \beta_i H_i \sim M$. To find (α_i, β_i) we shall represent Q_i, H_i and M in the above form. H_i is a non-exceptional fiber and can be represented as:

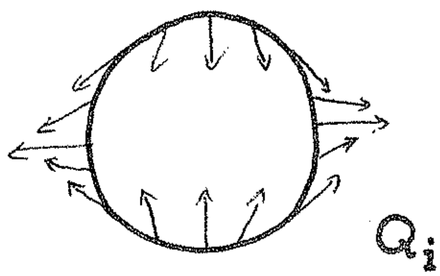


M can be represented as:

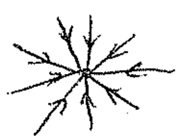


On F_2 there are four critical points of type:

at such a point Q_i is the curve:

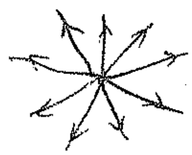


Thus $Q_i \sim M_i - H_i$, so $(\alpha_i, \beta_i) = (1,1)$ at such a point. There is one critical point of type:



Here $Q_i \sim M_i + H_i$, so $(\alpha_i, \beta_i) = (1,-1)$.

Finally, there is one critical point of type:

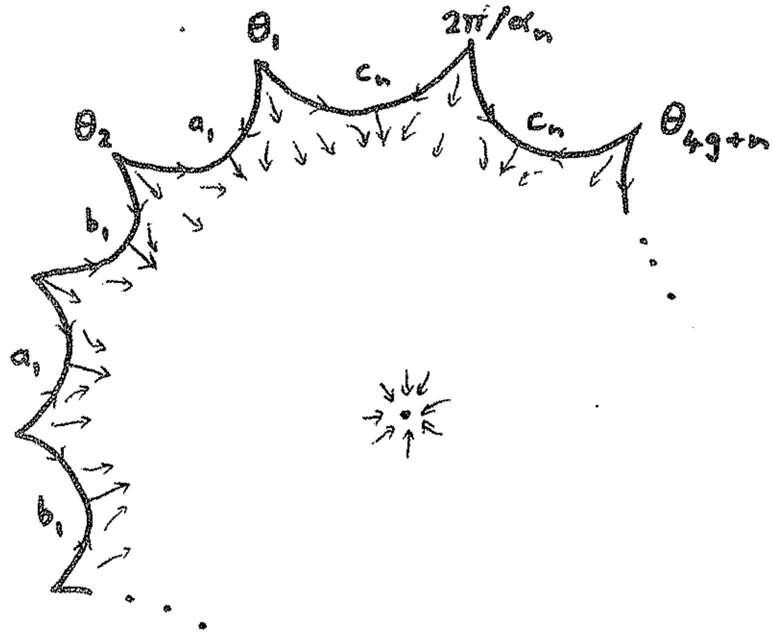


Here $(\alpha,\beta) = (1,-1)$.

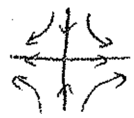
Thus $T^1F_2 = M(2;4(1,1),2(1,-1)) \cong M(2;(1,2))$.

The argument we have just given is the one which, in a more general form, was given by Hopf to prove his theorem that the Euler characteristic of any closed manifold is the sum of the indices of the zeroes of any vector field with isolated zeroes on that manifold.

Returning to the proof of our theorem, we can determine the (α_i,β_i) in exactly the same manner. Draw a fundamental domain for X/Γ as on page 60 and a vector field on that domain.



There are $(4g+2n)/2 + 2$ critical points:

$2g + n$ of type  with $(\alpha,\beta) = (1,1)$;

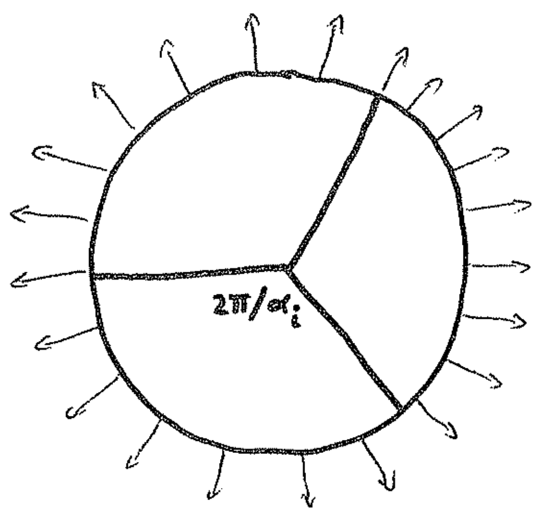
1 of type  with $(\alpha,\beta) = (1,-1)$;

1 of type  with $(\alpha,\beta) = (1,-1)$;

and for each $i = 1,\ldots,n$

1 of type  with $(\alpha,\beta) = (\alpha_i,-1)$.

Here the angle is $2\pi/\alpha_i$, and the fact that $(\alpha,\beta) = (\alpha_i,-1)$ follows from the following figure, which shows $\alpha_i Q_i = M_i + H_i$.



$\alpha_i Q_i$

Therefore $G/\Gamma = M(g;(2g+n)(1,1),2(1,-1),(\alpha_1,-1)\ldots(\alpha_n,-1))$

$= M(g;(1,2g-2),(\alpha_1,\alpha_1-1),\ldots,(\alpha_n,\alpha_n-1))$.

10. Seifert Groups

The aim of this section is to show that the Seifert invariant of $M = M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ is determined by $\pi_1(M)$. Along the way we define the notion of a Seifert group and classify these groups. Again for this section we assume if $g = 0$ then $n \geq 3$. We denote $\pi_1(M(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ by $\pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$.

Proposition 10.1. Either $(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)) = (1;\ )$ and $\pi(1;\ ) = \mathbb{Z}^3$, or $\langle h\rangle$ is the complete center $C(\pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ of $\pi = \pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$.

Proof. Recall we have the exact sequence

$$1 \to \langle h\rangle \to \pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)) \to \Gamma(g;\alpha_1,\ldots,\alpha_n) \to 1$$

Claim. $\Gamma = \Gamma(g;\alpha_1,\ldots,\alpha_n)$ has a trivial center unless

$$\begin{aligned}(g;\alpha_1,\ldots,\alpha_n) &= (1;\ ) \qquad (\Gamma = \mathbb{Z}^2)\\ &= (0;2,2,n) \quad (\Gamma = D_{2n}).\end{aligned}$$

Proof of Claim:

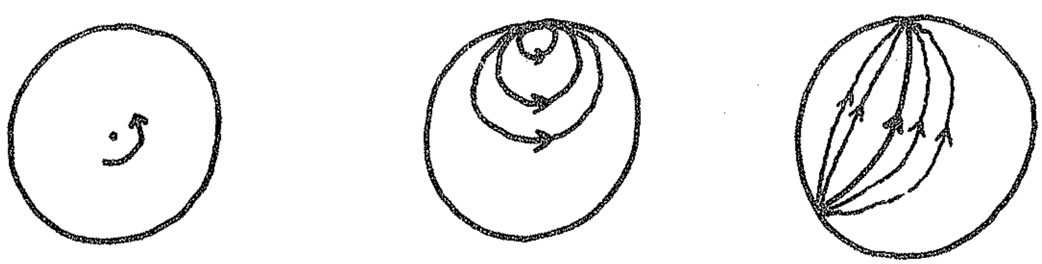
Case 1 Γ is hyperbolic

By the Brouwer fixed point theorem any $g \in \mathrm{Isom}^+(\mathbb{H}^2)$ has a fixed point in the extended plane $\overline{\mathbb{H}^2}$ (the closed disc in the Poincaré model). The elements of $\mathrm{Isom}^+(\mathbb{H}^2) \cong PSL(2,\mathbb{R})$ can be classified into three types according to how many fixed points each $g \in \mathrm{Isom}^+(\mathbb{H}^2)$ has in $\overline{\mathbb{H}^2} \cong D^2$ and where these fixed points are located. This classification can also be given in terms of the absolute value of the trace of g considered as an element of $SL(2,\mathbb{R})$. Let $1 \neq g \in \mathrm{Isom}^+(\mathbb{H}^2)$, then the three types of elements are:

a) g has a fixed point which is in the interior of D^2. In this case g is called an elliptic element and is a rotation about the fixed point with $|\mathrm{tr}g| < 2$.

b) g has just one fixed point and it is on the boundary of D^2. Here g is called a parabolic element and is rotations of horospheres. When g is parabolic $|\mathrm{tr}g| = 2$.

c) g has two fixed points on the boundary of D^2. g is called a hyperbolic element and is a translation along a geodesic and along curves of constant distance from this geodesic. Here $|\mathrm{tr}g| > 2$



elliptic parabolic hyperbolic

Since if $g_1, g_2 \in \mathrm{Isom}^+(X)$ and $g_1 g_2 = g_2 g_1$ then $g_1(X^{g_2}) = X^{g_2}$

(X^{g_2} is the fixed point set of g_2), we can conclude that any abelian subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$ is contained in a subgroup of the following type:

- S^1 - rotations about a given point;
- $\mathbb{R}$ - 1-parameter family of parabolic elements, with a given fixed point at infinity;
- $\mathbb{R}$ - translations along a given geodesic.

Hence in a discrete subgroup of $\mathrm{Isom}^+(\mathbb{H}^2)$ any abelian subgroup is cyclic. Thus if $c \in C(\Gamma)$ and $c \neq 1$ we can choose g not in the same 1-parameter subgroup. Then $cg \neq gc$ which is a contradiction. Therefore Γ has a trivial center.

Case 2: Γ is Euclidean

The only possible Euclidean subgroups are:

$\Gamma(2,4,4)$

$\Gamma(2,3,6)$

$\Gamma(3,3,3)$

$\Gamma(2,2,2,2)$

$\Gamma(1; \ )$.

If $\Gamma \neq \Gamma(1; \ )$, there exists elements of finite order in Γ, namely rotations about a point in $\mathbb{R}^2$. Assume $c \in C(\Gamma)$, then c fixes the fixed point of any such rotation. If x is a fixed point of $\gamma \in \Gamma$ then gx is a fixed point of $g\gamma g^{-1} \in \Gamma$, for any $g \in \Gamma$. Therefore

c fixes infinitely many points, so $c = \mathrm{id}$.

Case 3: Γ is spherical

The proof is just a case by case verification which we omit.

This completes the proof of the claim. The claim implies that $C(\pi) = \langle h \rangle$ except for possibly $\pi(1;)$ or $\pi(0;(2,1)(2,1),(n,\beta))$. These two cases can be checked individually (exercise), completing the proof of Proposition 10.1

Theorem 10.2. (spherical case) If $(\alpha_1,\alpha_2,\alpha_3)$ is one of $(2,2,n)$, $(2,3,3),(2,3,4)$ or $(2,3,5)$ then $\pi = \pi(0;(\alpha_1,\beta_1),(\alpha_2,\beta_2),(\alpha_3,\beta_3))$ determines the Seifert invariant (up to sign).

Proof. $\pi/C(\pi) = \Gamma(\alpha_1,\alpha_2,\alpha_3)$ which by Theorem 9.2 allows us to recover $\alpha_1,\alpha_2,\alpha_3$. Since $|\pi/[\pi,\pi]| = \alpha_1\alpha_2\alpha_3|e(M\to F)|$ we can recover $|e(M\to F)|$. It is sufficient to show $\alpha_1,\alpha_2,\alpha_3,e$ completely determine the Seifert invariant. That is, we must compute each $\beta_i \pmod{\alpha_i}$ in terms of $\alpha_1,\alpha_2,\alpha_3$, and e.

Case 1: $(\alpha_1,\alpha_2,\alpha_3) = (2,2,n)$. Recalling that the α_i and β_i are relatively prime, the $\beta_i \pmod{\alpha_i}$ are given by:

$$\beta_1 \equiv 1 \pmod 2, \quad \beta_2 \equiv 1 \pmod 2, \quad \beta_3 \equiv -ne \pmod n.$$

Case 2: $(\alpha_1,\alpha_2,\alpha_3) = (2,3,3)$. Up to exchanging β_2 and β_3 there are three possible cases:

a) $\beta_1 \equiv 1 \pmod 2$, $\beta_2 \equiv 1 \pmod 3$, $\beta_3 \equiv 1 \pmod 3$, $e \equiv \frac{5}{6} \pmod 1$

b) $\beta_1 \equiv 1 \pmod 2$, $\beta_2 \equiv 1 \pmod 3$, $\beta_3 \equiv 2 \pmod 3$, $e \equiv \frac{1}{2} \pmod 1$

c) $\beta_1 \equiv 1 \pmod 2$, $\beta_2 \equiv 2 \pmod 3$, $\beta_3 \equiv 2 \pmod 3$, $e \equiv \frac{1}{6} \pmod 1$.

These three possibilities are thus distinguished by the value of $e \pmod 1$.

The final two cases are completely analogous.

Case 3: $(\alpha_1, \alpha_2, \alpha_3) = (2,3,4)$

$\beta_1 \pmod 2$	$\beta_2 \pmod 3$	$\beta_3 \pmod 4$	$e \pmod 1$
1	1	1	$\frac{11}{12}$
1	1	3	$\frac{5}{12}$
1	2	1	$\frac{7}{12}$
1	2	3	$\frac{1}{12}$

Case 4: $(\alpha_1,\alpha_2,\alpha_3) = (2,3,5)$

β_1(mod 2)	β_2(mod 3)	β_3(mod 5)	e(mod 1)
1	1	1	$\frac{29}{30}$
1	1	2	$\frac{23}{30}$
1	1	3	$\frac{17}{30}$
1	1	4	$\frac{11}{30}$
1	2	1	$\frac{19}{30}$
1	2	2	$\frac{13}{30}$
1	2	3	$\frac{7}{30}$
1	2	4	$\frac{1}{30}$

Definition 10.3. A Seifert group $\pi = \pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ is a group with presentation:

$$\langle a_1,b_1,\ldots,a_g,b_g,q_1,\ldots,q_n,h \mid [a_i,h] = [b_i,h] = [q_j,h] = 1$$

$$i = 1,\ldots,g, j = 1,\ldots,n, q_j^{\alpha_j}h_j^{\beta_j} = 1, \prod_{j=1}^{n} q_j \prod_{i=1}^{g} [a_i,b_i] = 1\rangle.$$

Here we do not assume the (α_i,β_i) are relatively prime, but we do assume $\chi = 2-2g+\Sigma(\alpha_i-1)/\alpha_i$ is not positive.

Remark: As before, and by a similar argument we can show if π is a Seifert group and $\pi \neq \pi(1;\ ) = \mathbb{Z}^3$ then the center of π is $C(\pi) = \langle h \rangle$. Thus

$$1 \to \mathbb{Z} \to \pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)) \to \Gamma(g;\alpha_1,\ldots,\alpha_n) \to 1$$

is exact. By the theory of group extensions (see for example [Mac]), this extension is classified by an element $a \in H^2(\Gamma;\mathbb{Z})$. In our situation we can say more.

Theorem 10.4. If π,Γ and a are as above then $H^2(\Gamma;\mathbb{Z}) = \mathrm{Ab}\langle X_0,X_1,\ldots,X_n \mid \alpha_i X_i = X_0, i=1,\ldots,n\rangle$ and $a = \beta_1 X_1 + \cdots + \beta_n X_n$. Also any automorphism f of Γ induces $f^*\colon H^2(\Gamma;\mathbb{Z}) \to H^2(\Gamma;\mathbb{Z})$ of the following type: either $f^*(X_i) = X_{i'}$, for each i, where $i \mapsto i'$ is a permutation with $\alpha_i = \alpha_{i'}$, or $f^*(X_i) = -X_{i'}$, for each i, with $i \mapsto i'$ as before. We say $f\colon \Gamma \to \Gamma$ is "orientation preserving" or "orientation reversing" correspondingly.

Remark: In view of this theorem, the classifying element $a \in H^2(\Gamma;\mathbb{Z})$ is equivalent to the Seifert invariant, in that the one determines and is determined by the other (up to sign), so the Seifert invariant of a Seifert group is an abstract invariant of the group.

Observe also that if we tensor with $\mathbb{R}$ we have $H^2(\Gamma;\mathbb{R}) = H^2(\Gamma;\mathbb{Z}) \otimes \mathbb{R} \cong \mathbb{R}$ by $X_i \mapsto 1/\alpha_i \in \mathbb{R}$, and under this identification of $H^2(\Gamma;\mathbb{R})$ with $\mathbb{R}$, the classifying element a becomes the Euler number. This Euler number is still an invariant for Seifert groups, not just for Seifert manifolds.

Before proving Theorem 10.4, we need a brief summary of the theory of central extensions.

Fix an abstract group Γ, and an abelian group A. We define an equivalence relation on the set of short exact sequences of the form $1 \to A \to \pi \to \Gamma \to 1$ where $A \subseteq C(\pi)$ as follows: Given two short exact sequences

$$E_1: 1 \to A \to \pi_1 \to \Gamma \to 1, \quad E_2: 1 \to A \to \pi_2 \to \Gamma \to 1$$

we say $E_1 \sim E_2$ if there exists a group isomorphism $\varphi\colon \pi_1 \to \pi_2$ such that

$$\begin{array}{ccccccccc} 1 & \to & A & \to & \pi_1 & \to & \Gamma & \to & 1 \\ & & \| & & \downarrow{\scriptstyle\varphi} & & \| & & \\ 1 & \to & A & \to & \pi_2 & \to & \Gamma & \to & 1 \end{array}$$

commutes. We define $\mathrm{Ext}(\Gamma,A) = \{1 \to A \to \pi \to \Gamma \to 1 \,|\, A \subseteq \mathcal{C}(\pi)\}/\sim$, where $\sim$ is the equivalence relation just defined.

Given a homomorphism $f\colon \Gamma' \to \Gamma$ we define $f^*\colon \mathrm{Ext}(\Gamma,A) \to \mathrm{Ext}(\Gamma',A)$ by means of the following pullback diagram:

$$\begin{array}{ccccccccc} 1 & \to & A & \to & f^*\pi & \xrightarrow{\psi} & \Gamma' & \to & 1 \in \mathrm{Ext}(\Gamma',A) \\ & & \| & & {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle f} & & \\ 1 & \to & A & \longrightarrow & \pi & \xrightarrow{p} & \Gamma & \to & 1 \in \mathrm{Ext}(\Gamma,A) \end{array}$$

where $f^*\pi = \{(\pi,\gamma') \in \pi \times \Gamma' \,|\, p(\pi) = f(\gamma')\}$ and the maps $\varphi\colon f^*\pi \to \pi$ and $\psi\colon f^*\pi \to \Gamma'$ are the obvious projections onto the first and second coordinates respectively. Given a homomorphism $g\colon A \to A'$ we define $g_*\colon \mathrm{Ext}(\Gamma,A) \to \mathrm{Ext}(\Gamma,A')$ by means of the following pushout diagram:

$$\begin{array}{ccccccccc}
1 \longrightarrow & A & \longrightarrow & \pi & \longrightarrow & \Gamma & \longrightarrow 1 & \in \mathrm{Ext}(\Gamma,A) \\
 & g\big\downarrow & & \varphi\big\downarrow & & \big\| & & \\
1 \longrightarrow & A' & \longrightarrow & g_*\pi & \xrightarrow{\psi} & \Gamma & \longrightarrow 1 & \in \mathrm{Ext}(\Gamma,A')
\end{array}$$

where $g_*\pi = (\pi \times A')/\{(a,g(a)) \mid a \in A\}$ and φ,ψ are the obvious maps. Therefore what we have shown is that $\mathrm{Ext}(\Gamma,A)$ is a functor which is covariant in A and contravariant in Γ.

Let $E,E' \in \mathrm{Ext}(\Gamma,A)$ be represented by $1 \to A \to \pi \to \Gamma \to 1$ and $1 \to A \to \pi' \to \Gamma \to 1$ respectively. We define $E \times E' \in \mathrm{Ext}(\Gamma\times\Gamma, A\times A)$ to be the equivalence class represented by

$$1 \to A \times A \to \pi \times \pi' \to \Gamma \times \Gamma \to 1.$$

Define $\Delta\colon \Gamma \to \Gamma \times \Gamma$ by $\gamma \mapsto (\gamma,\gamma)$ and $\nabla\colon A \times A \to A$ by $(a,b) \mapsto a+b$. Then we let $E \oplus E' = \nabla_*\Delta^*(E\times E') = \Delta^*\nabla_*(E\times E')$. We leave it as an exercise to show that these two definitions of $E \oplus E'$ are the same and that this Baer sum turns $\mathrm{Ext}(\Gamma,A)$ into a group. It is a classical result that $\mathrm{Ext}(\Gamma,A) = H^2(\Gamma;A)$, see Maclane [Mac, p. 137] for a historical discussion. For our purposes, with $A = \mathbb{Z}$ we can take it as a definition of $H^2(\Gamma;\mathbb{Z})$.

Proof of Theorem 10.4. Recall $\pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n))$ and $\Gamma(g;\alpha_1,\ldots,\alpha_n)$ have presentations

$$\pi(g;(\alpha_1,\beta_1),\ldots,(\alpha_n,\beta_n)) = \langle a_1,b_1,\ldots,a_g,b_g,q_1,\ldots,q_n,h \mid$$

$$[a_i,h] = [b_i,h] = [q_j,h] = 1,\quad q_j^{\alpha_j} h^{\beta_j} = 1,$$

$$i = 1,\ldots,g, j = 1,\ldots,n, \prod_{j=1}^{n} q_j \prod_{i=1}^{g} [a_i,b_i] = 1\rangle$$

$$\Gamma(g;\alpha_1,\ldots,\alpha_n) = \langle a_1,b_1,\ldots,a_g,b_g,q_1,\ldots,q_n \mid q_j^{\alpha_j} = 1, j = 1,\ldots,n$$

$$\prod_{j=1}^{n} q_j \prod_{i=1}^{g} [a_i,b_i] = 1\rangle.$$

Let

$$1 \to \mathbb{Z} \to \pi \to \Gamma \to 1$$

be a representative of an element of $\mathrm{Ext}(\Gamma,\mathbb{Z})$. If we choose lifts $\bar{a}_i$, $\bar{b}_i$, $\bar{q}_j$ of a_i, b_i, q_j $i = 1,\ldots,g$, $j = 1,\ldots,n$ respectively, it is easily seen that π has presentation

$$\langle \bar{a}_1,\bar{b}_1,\ldots,\bar{a}_g,\bar{b}_g,\bar{q}_1,\ldots,\bar{q}_n,h \mid [\bar{a}_i,h] = [\bar{b}_i,h] = [\bar{q}_j,h] = 1,$$

$$\bar{q}_j^{\alpha_j} = h^{s_j}, i = 1,\ldots,g, j = 1,\ldots,n, \prod_{j=1}^{n} \bar{q}_j \prod_{i=1}^{s} [\bar{a}_i,\bar{b}_i] = h^{s_0}\rangle$$

for some integers $s_0,\ldots,s_n$. We denote this presentation by $\pi(s_0,\ldots,s_n)$. We denote the element of $\mathrm{Ext}(\Gamma,\mathbb{Z})$ represented by

$$1 \to \mathbb{Z} \to \pi(s_0,\ldots,s_n) \to \Gamma \to 1$$

by $E(s_0,\ldots,s_n)$. Notice that we made a choice for the lifts of a_i, b_i, q_j. We could have chosen different lifts $\bar{\bar{a}}_i = \bar{a}_i h^{\ell_i}$,

$\bar{\bar{b}}_i = \bar{b}_i h^{m_i}$, $\bar{\bar{q}}_j = \bar{q}_j h^{k_j}$ in which case π would have presentation

$$<\bar{\bar{a}}_1,\bar{\bar{b}}_1,\ldots,\bar{\bar{a}}_g,\bar{\bar{b}}_g,\bar{\bar{q}}_1,\ldots,\bar{\bar{q}}_n,h \mid [\bar{\bar{a}}_i,h] = [\bar{\bar{b}}_i,h] = [\bar{\bar{q}}_j,h] = 1,$$

$$\bar{\bar{q}}_j^{\alpha_j} = h^{s_j+\alpha_j k_j}, i = 1,\ldots,g, j = 1,\ldots,n,$$

$$\prod_{j=1}^{n} \bar{\bar{q}}_j \prod_{i=1}^{g} [\bar{\bar{a}}_i,\bar{\bar{b}}_i] = h^{s_0+k_1+\cdots+k_n}>$$

Thus any element $E = E(s_0,\ldots,s_n) \in \mathrm{Ext}(\Gamma,\mathbb{Z})$ corresponds to an element $[s_0,\ldots,s_n] \in \mathbb{Z}^{n+1}/I$ where $I = \{(k_1 + \cdots + k_n, \alpha_1 k_1, \cdots, \alpha_n k_n) \mid k_i \in \mathbb{Z}\} =$ span $\{(1,\alpha_1,0,\ldots,0),(1,0,\alpha_2,0,\ldots,0),\ldots,(1,0,\ldots,0,\alpha_n)\}$. Therefore we have an injective map $F\colon \mathrm{Ext}(\Gamma,\mathbb{Z}) \to \mathbb{Z}^{n+1}/I$. We leave as an exercise the proof that $E(s_0,\ldots,s_n) \oplus E(s_0',\ldots,s_n') = E(s_0+s_0',\ldots,s_n+s_n')$ and hence F is a group homomorphism.

$\mathbb{Z}^{n+1}/I = <X_0,X_1,\ldots,X_n \mid \alpha_i X_i = X_0, i = 1,\ldots,n>$. (This can be seen by representing $X_0 = (1,0,\ldots,0)$, $X_1 = (0,-1,0,\ldots,0),\ldots, X_n = (0,\ldots,0,-1)$). Therefore to see $H^2(\Gamma;\mathbb{Z})$ has the desired form we must show F is onto.

To this end it suffices to show that given $[s_0,\ldots,s_n] \in \mathbb{Z}^{n+1}/I$ we get $E(s_0,\ldots,s_n) \in \mathrm{Ext}(\Gamma,\mathbb{Z})$. This requires showing that given $[s_0,\ldots,s_n]$ the kernel of the map $p\colon \pi(s_0,\ldots,s_n) \to \Gamma$ is $\mathbb{Z}$ and not a non-trivial quotient of $\mathbb{Z}$. We define a function $\nu\colon \mathbb{Z}^{n+1}/I \to S(\mathbb{Z}) =$ {subgroups of $\mathbb{Z}$} by $[s_0,\ldots,s_n] \mapsto \ker(\mathbb{Z} \to \pi(s_0,\ldots,s_n))$. Thus

(**) $$1 \to \mathbb{Z}/\nu([s_0,\ldots,s_n]) \to \pi(s_0,\ldots,s_n) \to \Gamma \to 1$$

is exact. Note ν has the following properties:

1) $\nu(k[s_0,\ldots,s_n]) = k\nu([s_0,\ldots,s_n])$ for all $k \in \mathbb{Z}$

2) $\nu([s_0,\ldots,s_n] + [s'_0,\ldots,s'_n]) \subseteq \nu([s_0,\ldots,s_n]) + \nu([s'_0,\ldots,s'_n])$

for any $[s_0,\ldots,s_n],[s'_0,\ldots,s'_n] \in \mathbb{Z}^{n+1}/I$.

Since (**) is exact, if we can show the image of ν is trivial, we will have shown $H^2(\Gamma;\mathbb{Z}) = \langle X_0,X_1,\ldots,X_n \mid \alpha_i X_i = X_0, i = 1,\ldots,n\rangle$. Note that for every $[s_0,\ldots,s_n] \in \mathbb{Z}^{n+1}/I$ there exists $k,m \in \mathbb{Z}$ such that $k[s_0,\ldots,s_n] = mX_0$. Therefore by property 1 it is sufficient to show $\nu(mX_0) = \langle 0\rangle$. In fact it is sufficient to show $\nu(\beta_1X_1 + \cdots + \beta_nX_n) = \langle 0\rangle$ for some $\beta_1X_1 + \cdots + \beta_nX_n$ with $\sum_{i=1}^{n} \beta_i/\alpha_i \neq 0$ since $\alpha_1\cdots\alpha_n(\beta_1X_1+\cdots+\beta_nX_n) = mX_0$ with $m \neq 0$. In the hyperbolic case we saw $(2g-2)X_0 + \sum_{i=1}^{n} ((\alpha_i - 1)/\alpha_i)X_i$ classifies $(T^1\mathbb{H})/\Gamma$ and

$$\begin{array}{ccccccccc} 1 \to & \pi_1(T^1\mathbb{H}) & \to & \pi_1(T^1\mathbb{H}/\Gamma) & \to & \Gamma & \to 1 \\ & \| & & \| & & \| & \\ & \mathbb{Z} & \longrightarrow & \pi & \longrightarrow & \Gamma & \end{array}$$

is exact. Hence $\nu((2g-2)X_0 + \sum_{i=1}^{n} ((\alpha_i-1)/\alpha_i)X_i) = \langle 0\rangle$.

For the Euclidean case we recall that an earlier computation showed the universal abelian covers of the Seifert manifolds

$M(0;(2,\beta_1),(3,\beta_2),(6,\beta_3))$

$M(0;(2,\beta_1),(4,\beta_2),(4,\beta_3))$

$M(0;(3,\beta_1),(3,\beta_2),(3,\beta_3))$

$M(0;(2,\beta_1),(2,\beta_2),(2,\beta_3),(2,\beta_4))$

are

$M(1;(1,1))$

$M(1;(1,2))$

$M(1;(1,3))$

$M(1;(1,4))$

respectively, provided we assume $e(M \to F) < 0$ in each case. That is the universal abelian cover is the genuine S^1-bundle over T^2 of Euler number $-1,-2,-3$ or -4. Thus h has infinite order in the fundamental group of the cover and consequently in the fundamental group of the Seifert manifold in question.

It remains to show that automorphisms γ of Γ induce isomorphisms γ^* of $H^2(\Gamma;\mathbb{Z})$ of the desired type. As we saw in the proof of Theorem 9.2, γ must map $\langle q_j \rangle$ to a conjugate of some $\langle q_{j'} \rangle$ with $\alpha_j = \alpha_{j'}$. For simplicity of notation we assume $j = j'$ and hence $\gamma(q_j) = g_j^{-1}(q_j^{m_j})g_j$ with $m_j \in \mathbb{Z}$ g.c.d. $(m_j,\alpha_j) = 1$ and $g_j \in \Gamma$. Consider the action induced by γ on $\pi(s_0,\ldots,s_n)$. $\gamma(q_j^{\alpha_j}) = g_j^{-1}h^{-s_j m_j}g_j = h^{s_j m_j}$. Consequently for the element X_j of $H^2(\Gamma;\mathbb{Z})$ we have $\gamma^*(X_j) = m_j X_j$. γ^* is an isomorphism hence we must have $m_j = \pm 1$.

Remark: Much of the above proof can be put in a more general setting. If we let $\Gamma = \langle X_1,\ldots,X_g | W_1 = 1,\ldots,W_r = 1\rangle$ and A be an abelian group, then by a completely analagous construction as in the proof of Theorem 10.4 we obtain a map $F\colon \mathrm{Ext}(\Gamma,A) \to A^r/I$ where $I = \mathrm{Im}(W\colon A^g \to A^r)$ and $W = (W_1^{ab},\ldots,W_r^{ab})$ (W_i^{ab} is the abelianization of W_i). More precisely: given

$$1 \to A \to E \to \Gamma \to 1$$

a representative of an element of $\mathrm{Ext}(\Gamma,A)$, E has presentation:

$$(*) \qquad \langle \overline{X}_1,\ldots,\overline{X}_n, \text{"}A\text{"} | \overline{W}_1 = a_1,\ldots,\overline{W}_r = a_r, \text{"relations in } A\text{"}, \text{"}A \text{ central"}\rangle$$

(Here by "A" we mean a set of generators of A, by "relations in A" we mean a set of defining relations of A and by "A central" we mean A is central in E.) As before $\overline{X}_i$, $\overline{W}_i$ are lifts of X_i and W_i respectively. If we had chosen different $\overline{X}_i$'s say $\overline{\overline{X}}_i = \overline{X}_i a_i$ we would have $\overline{W}_i(\overline{X}_1 a_1,\ldots,\overline{X}_n a_n) = \overline{W}_i(\overline{X}_1,\ldots,\overline{X}_n) W_i^{ab}(a_1,\ldots,a_n)$. Thus by letting $F(E) = (a_1,\ldots,a_r)$ we get a well-defined map $F\colon \mathrm{Ext}(\Gamma,A) \to A^r/\mathrm{Im}(W_1^{ab},\ldots,W_r^{ab})$.

We define a map $\nu\colon A^r/I \to S(A) = \{\text{subgroups of } A\}$ by $[a_1,\ldots,a_r] \mapsto \ker(A \to \pi(a_1,\ldots,a_r))$ where $\pi(a_1,\ldots,a_r)$ is the group presented as in (*) above.

Hence $1 \to A/\nu([a_1,\ldots,a_r]) \to \pi(a_1,\ldots,a_r) \to \Gamma \to 1$ is exact. ν as defined in this more general way still satisfies properties 1) and 2) as listed in the proof of Theorem 10.4. Therefore in general, given a presentation of Γ and an abelian group A we get a subgroup I of A^r and a map $\nu\colon A^r/I \to S(A)$ such that $\mathrm{Ext}(\Gamma,A) = \nu^{-1}(0)$ and $\mathrm{Ext}(\Gamma,A/nA) = \nu^{-1}(nA)\ /n(A^r/I)$.

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