Smooth Circle Actions on 3-Manifolds

We want to create a fairly large family of compact oriented smooth manifolds M^3 which admit fixed point free actions of $S^1 =$ unit complex numbers.

Smooth group action = smooth map

$\Phi : G \times M \longrightarrow M$ G compact Lie

s.t. if $g \cdot x = \Phi(g, x)$, then

$g_1(g_2 x) = (g_1 g_2) x$, $1x = x$.

Each map $L_g(x) = g \cdot x$ is a diffeomorphism and $(L_g)^{-1} = L_{g^{-1}}$.

Fixed point set = all $x \in M$ s.t. $gx = x$ for each $g \in G$. Write as M^G. Fixed point free $\Longleftrightarrow M^G = \phi$.

Isotropy subgroup at $x \in M$ is all $g \in G$ s.t. $gx = x$, written G_x SUBSCRIPT
The orbit of x under G, written $G \cdot x$ is the set of all points $g \cdot x$ where g runs through the elements of G. $G \cdot x \cong G/G_x$ (in fact smoothly — when $G = S^1$ or G is finite one can see this directly).

Basic examples are principal S^1-bundles = unit circle bundles in complex line bundles.
However, there are others: S^3 = unit sphere in $\mathbb{C}^2$.
$S(t^a + t^b) = S^3$ with the circle action
$z(w_1, w_2) = (w_1^a, w_2^b)$ where a & b are

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relatively prime.

Orbit structure $S^1/\mathbb{Z}_a \times \{1\} \subseteq T^2 \subseteq S^3$ (T^2 = all (w_1, w_2) with $|w_1| = |w_2|$).
$\{1\} \times S^1/\mathbb{Z}_b \subseteq T^2 \subseteq S^3$. All other orbits are free; i.e. the isotropy subgroups are $\{1\}$.

Note $S(t^a+t^b) \cong S^2$. To see this, consider the following commutative diagram in which the vertical arrows are orbit space projections: [a, b rel. prime]

$$\begin{array}{ccc} S(t^a+t^b) & \xrightarrow{f} & S(t^{ab}+t^{ab}) \\ \downarrow & & \downarrow \\ S(t^a+t^b)/S^1 & \xrightarrow{\bar{f}} & S(t^{ab}+t^{ab})/S^1 \\ & & \cong S^2. \end{array}$$

Here $f(w_1, w_2) = \dfrac{1}{\sqrt{|w_1|^{2b}+|w_2|^{2a}}}(w_1^b, w_2^a)$

It is a straightforward elementary (but slightly messy) exercise to prove that $\bar{f}$ is 1-1 onto.

ANOTHER EXAMPLE The Poincaré homology 3-sphere SO_3/A_5.

Generalities G = connected Lie group
H = finite subgroup, so $G \longrightarrow G/H$ is a regular covering space projection (in fact, smooth). Let $K \subseteq G$ be a closed subgroup and let K act on G/H by $k \cdot gH = (kg)H$.

What is the isotropy subgroup K_{gH}?

$kgH = gH \iff g^{-1}kg \in H$, so the isotropy subgroup is $K \cap gHg^{-1}$.

Suppose now that $G = SO_3$, $H = A_5$, $K = SO_2 \cong S^1$. Every element of A_5 has order 1, 2, 3 or 5, ~~[illegible]~~ ~~[illegible]~~ and every finite subgroup of S^1 is cyclic. Therefore the only possible isotropy subgroups for the SO_2 action on $\Sigma^3 = SO_3/A_5$ are the cyclic subgroups of order 1, 2, 3, 5.

Now every element of SO_3 is conjugate to a plane rotation, and this means that every element lies in a conjugate of SO_2. If we apply this to elements of A_5, we see that we can find group elements $g \in SO_3$ such that $H \cap g^{-1}Kg$ is an arbitrary cyclic subgroup of H, or equivalently there are group elts. g such that $K \cap gHg^{-1}$ has order 1, 2, 3, 5 (for suitable, distinct choices — not the same g for all!).

<u>CLAIM:</u> There is only <u>one</u> orbit with isotropy subgroup $\mathbb{Z}_a$ where $a = 2, 3, 5$.

Every orbit with isotropy subgroup $\mathbb{Z}_a$ corresponds to some $g \in G [= SO_3]$ such that $H \cap g^{-1}Kg$ has order equal to a. Suppose we are given g_1, g_2 s.t. $L_i = H \cap g_i^{-1}Kg_i$ have order a [equivalently, the same for $K \cap g_i H g_i^{-1}$].

Then L_1 and L_2 are conjugate in H, so let $h_0 \in H$ be such that $h_0^{-1} L_1 h_0 = L_2$. Then we have $H \cap g_2^{-1} K g_2 = H \cap h_0^{-1} g_1^{-1} K g_1 h_0$. In particular, this means that

$$g_2^{-1} K g_2 \cap h_0^{-1} g_1^{-1} K g_1 h_0 \neq \{1\}.$$

and since conjugates of SO_2 are equal if their intersections are more than $\{1\}$, it follows that $g_2^{-1} K g_2 = h_0^{-1} g_1^{-1} K g_1 h_0$. The latter implies that $g_2 (g_1 h_0)^{-1}$ lies in the normalizer of K.

By the structure of H inside SO_3 we know that there is an element $h_1 \in N(g_2^{-1} K g_2) \cap H$ with $h_1 \notin g_2^{-1} K g_2 \cap H$, and it follows that $g_2 h_1 g_2^{-1}$ lies in $N(K)$ but not in K. Note that $N(K)/K \cong \mathbb{Z}_2$.

We are trying to prove that the orbits Kg_1H and Kg_2H are equal. There are two cases.

Case (i) $g_2(g_1h_0)^{-1} \in K$ — Then $k=(g_1h_0)g_2^{-1} \in K$ too, so we have $kg_2H = g_1h_0H = g_1H$, so $Kg_2H = Kg_1H$.

Case (ii) $g_2(g_1h_0)^{-1} \in N(K)$ but not K. Let h_1 be as above. Then we must have

$$(g_2h_1g_2^{-1})g_2(g_1h_0)^{-1} \in K, \text{ so that}$$

$$\text{"} (g_2h_1)(g_1h_0)^{-1}$$

$k' = (g_1h_0)(g_2h_1)^{-1} \in K$ too. We can now argue as in Case (i) to show that $Kg_1H = Kg_2H$.