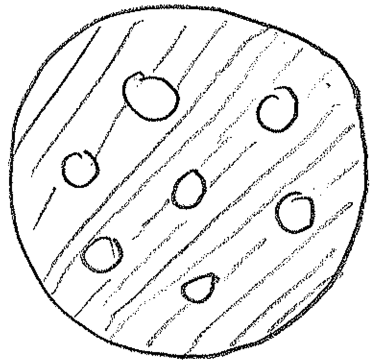
Seifert fiberings over S^2

Emphasis on construction rather than classification.

Start out with a 2-disk with holes removed

$E(g)$



Suppose $g \geq 0$ holes are removed

Then $\pi_1(E_g)$ is a free group on g generators

$$E(g) \cup_{id} E(g) \cong \#^g T^2 \text{ if } g \geq 1.$$

untwisted double

$$\partial E(g) = \underset{\text{outer circle}}{C_0} \amalg - \underset{\text{inner circles.}}{(C_1 \amalg \ldots \amalg C_g)}$$

Start with $E(g) \times S^1$, where S^1 acts by translation on the second factor, trivial action on the first.

We want to glue on pieces to each $C_i \times S^1$ such that

(0) The piece attached along $C_0 \times S^1$ has no "singular fibers" — the isotropy subgroups are always $\{1\}$.

(i) The piece attached along $C_i \times S^1$ has <u>one</u> "singular fiber" with isotropy subgroup $\mathbb{Z}_{a_i}$ for some $a_i > 1$.

How to attach the pieces?

For each a_i choose b_i relatively prime to a_i, and consider

$$S^1 \times_{\mathbb{Z}_{a_i}} D(t^{b_i}) =$$

unit disk

$S^1 \times D(t^{b_i})$ modulo $(wz, v) \sim (w, zv)$.

[balanced product construction]

$$D(t^b) \subseteq S^1 \times_{\mathbb{Z}_a} D(t^b) \longrightarrow S^1/\mathbb{Z}_a$$

S^1-equivariant complex line bundle

Note A G-vector bundle over a space X with G-action is a vector bundle $\pi: E \longrightarrow X$ s.t. E has a G-action with the following property: g takes the fiber E_x to E_{gx} by a linear isomorphism.

There are some important alternate descriptions of these balanced products:

Prop. 1 Suppose G acts on X and $H \subseteq G$. Then $G \times_H X \underset{G}{\cong} G/H \times X$.

Proof The map is a "clear isomorphism" which sends $[g, x]$ to (gH, gx), and of course a crucial point is to verify that it is well-defined.

Two points (g, x) and (g', x') are equivalent $\Leftrightarrow$ there is some $h \in H$ such that $(gh, h^{-1}x) = (g', x')$. Note that the latter implies $(gh, x) \sim (ghh^{-1}, hx) = (g, hx)$; in fact, both formulations yield the same quotient.

So suppose now that we have (g, x) and $(gh, h^{-1}x)$ which go to (gH, gx) and $(\underset{\substack{\shortparallel \\ gH}}{ghH}, \underset{\substack{\shortparallel \\ gx}}{(gh)h^{-1}x})$ respectively.

Therefore the shear map is well-defined. An explicit inverse is given by sending (gH, y) to $[g, g^{-1}y]$; one can check this is also well-defined.

[The preceding works in any reasonable category]

Here is a more specialized result:

<u>Prop. 2</u> Let a, b be relatively prime. Then $S(t^a) \times S(t^b) \underset{S^1}{\cong} S(t^1) \times \underbrace{|S^1|}_{\text{trivial action}}$.

<u>Proof</u> Choose integers p + r s.t. $ap + br = 1$, and define

$$\varphi : S(t^a) \times S(t^b) \longrightarrow S(t) \times |S^1|$$

$$\varphi(x, y) = (x^p y^r, x^{-b} y^a).$$

We need to check φ is S^1-equivariant.

$$\begin{tikzcd}
(x, y) \arrow[r] \arrow[d] & (x^p y^r, x^{-b} y^a) \arrow[d] \\
(z^a x, z^b y) \arrow[r] & (z^{ap} x^p z^{br} y^r, z^{-ba} x^{-b} z^{ab} y^a)
\end{tikzcd}$$

$$= (z^{ap+br} x^p y^r, z^{ab-ba} x^{-b} y^a)$$

The last pair is ~~$(z x^p y^r, x^{-b} y^a)$~~ ~~because~~