Computing the abelianization $H_1(M;\mathbb{Z})$ is relatively easy. We have a free abelian group with generators $Y_1, \dots, Y_g, Z$; the relations to be factored out are $a_j Y_j = -b_j Z$ and $\sum_i Y_i = -f Z$. So we want the cokernel of the following matrix

$$\begin{pmatrix} a_1 & & & & -b_1 \\ & \ddots & & 0 & \vdots \\ & & \ddots & & \vdots \\ & 0 & & a_g & -b_g \\ 1 & 1 & \cdots & 1 & -f \end{pmatrix}$$

Clearly it has rank at least $g$, so it is either finite or finite $\oplus\, \mathbb{Z}$. To determine which, we need to compute its determinant. But the latter can be found using row operations to kill the 1's in the bottom row except in the corner.

This yields $\left(\sum \frac{b_i}{a_i} - f\right) a_1 \ldots a_g$.

Hence $H_1$ is finite $\Leftrightarrow \sum \frac{b_i}{a_i} - f = 0$.

In these cases $M$ is a <u>rational</u> homology 3-sphere. The manifolds obtained in this fashion include all <u>lens spaces</u> $S(t^a + t^b)/\mathbb{Z}_k$ where $k$ is prime to $a$ and $b$, so that $S(t^a+t^b) \to S(t^a+t^b)/\mathbb{Z}_k$ is a covering space proj.

<u>Note on lens spaces</u> $L(a,b;k)$. $k$ fixed

$\left\{\begin{matrix}\text{Homeomorphism}\\ \text{Diffeomorphism}\end{matrix}\right\}$ classification $b/a \equiv \pm b'/a' \ (k)$

Homotopy classification $b/a \equiv \pm c^2 b'/a' \ (k)$.

<u>Examples</u> $L(1,1;5) \not\simeq L(1,2;5)$

$L(1,1;7) \simeq L(1,2;7)$ but $L(1,1;7) \not\approx L(1,2;7)$.

One obtains an integral homology sphere $\Leftrightarrow$

$$\left(\sum \frac{b_i}{a_i} - f\right) = \frac{1}{a_1 \dots a_g}.$$

Cor. Let $a_1, \dots, a_n$ be a sequence of integers which are relatively prime. Then there is a manifold $\Sigma^3$ with the following properties:

(i) $\Sigma^3$ is an integral homology 3-sphere

(ii) $\Sigma^3$ has a smooth $S^1$ action with $g$ exceptional orbits whose isotropy subgroups have orders $a_1, \dots, a_g$.

Note. All but finitely many of these groups have an infinite central subgroup. One gets a simply connected manifold $\Leftrightarrow g \leq 2$.

## Proof of Corollary

It suffices to find integers $b_1, \ldots, b_g$ s.t. $\sum \frac{b_i}{a_i} = 1$. This is a straightforward algebraic exercise.