Existence of Tubular Neighborhoods

Locally, we know how a smooth submanifold N^n sits inside its ambient manifold M^m (where $n < m$), for we can change variables smoothly so that the inclusion of the submanifold locally looks like the flat inclusion of $\mathbb{R}^n$ in $\mathbb{R}^m$ as the subspace of all vectors whose last $m - n$ coordinates are all zero. For many purposes it is useful, or even absolutely essential, to have a comparable global picture.

<u>Easy example.</u> Take S^n as a submanifold of $\mathbb{R}^{n+1}$ in the usual way. Then one knowns that the open neighborhood $\mathbb{R}^{n+1} \setminus \{0\}$ of S^n is diffeomorphic to the product $S^n \times \mathbb{R}$ by the map sending $\mathbf{v}$ to the pair $(\,|\mathbf{v}|^{-1}\mathbf{v}, \ln|\mathbf{v}|\,)$, and under this diffeomorphism S^n corresponds to $S^n \times \{0\}$.

<u>Easy generalizations.</u> Look at regular level hypersurfaces of $\mathbb{R}^{n+1}$ defined by equations of the form $f(x) = 0$, where f is a smooth real valued function with the basic regularity property one first sees in the theory of Lagrange multipliers f: If $f(x) = 0$ then the gradient $\nabla f(x)$ is nonzero. In this case, the zero set N of f is a smooth $n-$dimensional submanifold of $\mathbb{R}^{n+1}$ and the normal thickening map

$$Q : N \times \mathbb{R} \longrightarrow \mathbb{R}^{n+1}$$

given by $Q(x, t) = x + t \cdot \nabla f(x)$ is a smooth map such that for each x the derivative matrix/linear transformation $DQ(x, 0)$ is invertible (this uses the fact that the gradient is nonzero perpendicular to the hypersurface everywhere). This means that for each x there is some $h > 0$ and some neighborhood U of x in N such that the restriction of Q to $N \times (-h, h)$ is mapped diffeomorphically onto a neighborhood of x in $\mathbb{R}^{n+1}$. As the picture below suggests, the map Q will usually not define a diffeomorphism on all of $N \times \mathbb{R}$ (in contrast to the case of the sphere). However, one can use some point set topology to show that there is an open neighborhood V of $N \times \{0\}$ such that Q maps V diffeomorphically onto a neighborhood of N in $\mathbb{R}^{n+1}$ (in the picture, this neighborhood is indicated by the red line segments; see also the second picture below).

The preceding examples might lead one to ask if for every submanifold inclusion of some N^n in an M^m there is a neighborhood diffeomorphic to a product manifold $N^n \times \mathbb{R}^{m-n}$. However, this is false; the simplest example of this sort is given by taking M to be a Möbius strip and N to be the circle in the center of the

Möbius strip. The well – known "one – sidedness" property of the Möbius strip then leads to the following conclusion:

If V is an open neighborhood of N in M and x is a point of N, then there is an open neighborhood W of x in V and all points of $W \setminus N$ lie in the same connected component of $M \setminus N$.

This could not happen if N had a neighborhood U which was homeomorphic to $N \times \mathbb{R}$ (why?).

On the other hand, the Möbius strip example also provides a key to understanding what does happen in general. This may be seen using a standard parametrization of the Möbius strip as a ruled surface, in which every point lies on a unique line segment passing through the central circle N:

$$\sigma(u, v) = (\cos u, \sin u, 0) + v \cdot \big(\cos u \cdot \cos(u/2), \sin u \cdot \cos(u/2), \sin(u/2)\big)$$

One important feature of this parametrization is the existence of a projection from M to N given geometrically by taking all points on the line segment in M through x and mapping them to x; this projection is a smooth submersion whose restriction to N is the identity map.

In fact, using a suitable change of coordinates it is possible to think of the line segments through points as 1 – dimensional vector spaces, each with its own addition and scalar multiplication, and with the zero given by the unique point of N through which the line segment passes. This is an example of a vector bundle structure on M over N.

The **Tubular Neighborhood Theorem** states that every submanifold N in M has a neighborhood with such a vector bundle structure, and in fact this bundle structure is unique up to a suitable notion of equivalence. This result is important for many reasons, some of which are related to the geometry of M and N, and others of which expedite the use of algebraic topology and homotopy theory for studying smooth manifolds. One reference for the statements and proofs of the basic existence theorems is Bredon, ***Geometry and Topology***, specifically pages 93 – 94 (Theorem **II.11.4**) and 99 – 100 (Theorem **II.11.14**). These proofs are brief and direct, and they do not use vector bundles explicitly. Since the formulations and proofs of the uniqueness theorems for tubular neighborhoods require vector bundle structure, it will be worthwhile to reformulate some of Bredon's exposition in terms of vector bundles. We shall refer to the following old Mathematics 205C notes for the basic definitions for such objects:

http://math.ucr.edu/~res/math260s10/coursenotes205C.pdf

More precisely, the general concept of a vector bundle is discussed beginning in Subsection V.1.3 on page 173 (the preceding subsections in V.1 discuss basic examples of vector bundles on real and complex projective spaces; one can skip the discussion of such examples and proceed straight to the last two paragraphs of the subsection without loss of continuity). Subsections V.1.4 through V.1.6 give most of the formal definitions and foundational results, and Subsection V.2.1 gives an important construction called the ***pullback***. We shall freely use material from these subsections henceforth.

In the theory of vector bundles, it is often useful and sometimes essential to put inner products on the vector spaces which lie over the points of the base space. As in the case of tangent bundles, such structures are called Riemannian metrics, and Subsection V.2.4 of the notes (beginning on page 184) gives the definitions and main results, including the all – important theorems about the existence of such structures on arbitrary vector bundles.

We shall need one additional concept involving vector bundles; namely, the notion of a ***vector sub – bundle***. As for vector bundles, there are both topological and smooth versions of this concept.

<u>Definition.</u> Let $\alpha: A \to X$ and $\beta: B \to X$ be (topological or smooth) vector bundles over X, for each x in X let A_x and B_x denote the vector spaces (fibers) in A and B which map to x under the projection maps, and let $\mathbf{T}: A \to B$ be a (continuous or smooth) mapping such that for each x in X the mapping $\mathbf{T}$ sends A_x to B_x by a linear monomorphism. Then we shall say that the image $\mathbf{T}[A]$ is a (continuous or smooth) ***vector sub – bundle*** of B. In the smooth category, it follows that the mapping $\mathbf{T}$ defines a smooth embedding (onto a smooth submanifold of B).

<u>Fundamental Examples.</u> **1.** Suppose that X is a nowhere zero smooth vector field on the smooth manifold M. Then there is a vector bundle inclusion from the product bundle $M \times \mathbb{R}$ to $\mathbf{T}(M)$ sending (y, t) to $t \cdot X(y)$.

2. Suppose that N^n is a smooth embedding of M^m (where $n < m$) and j is the inclusion mapping. Then the map of tangent spaces $\mathbf{T}(j)$ from $\mathbf{T}(N^n)$ to $\mathbf{T}(M^m)$ identifies $\mathbf{T}(N^n)$ a subbundle of $\mathbf{T}(M^m)$ restricted to the submanifold N^n.

Predictably, the second example is particularly important to us in the present discussion. In any case, we have the following fundamentally important result, which is basically a general form of Proposition **II.11.2** on page 93 of Bredon:

Complementary Subbundle Theorem. Suppose we are given the data needed for the definition of a (topological or smooth) sub – bundle, and use the notation in the definition above; assume also that we are given a (continuous or smooth) Riemannian metric on $\boldsymbol{B}$. For each $\boldsymbol{x}$ in $\boldsymbol{X}$ let $\boldsymbol{C_x} \subset \boldsymbol{B_x}$ be the orthogonal complement of $\mathbf{T}[\boldsymbol{A_x}]$ in $\boldsymbol{B_x}$, and let $\boldsymbol{C}$ be the union of these vector subspaces. Then the restriction of $\boldsymbol{\beta}$ to $\boldsymbol{C}$ defines a (topological or smooth) vector bundle which is a sub – bundle of $\boldsymbol{B}$; furthermore, in the smooth category this sub – bundle is also a smooth submanifold.

Less formally, ***for each sub – bundle there is a complementary sub – bundle***, and over each point the fibers of these subbundles are complementary subspaces (their linear sum is the whole fiber, and their intersection only contains the zero vector).

Proof. Let $\boldsymbol{x}$ be a point in $\boldsymbol{X}$, and let $\boldsymbol{U}$ be an open neighborhood of $\boldsymbol{x}$ such that the restrictions of both $\boldsymbol{A}$ and $\boldsymbol{B}$ to $\boldsymbol{U}$ are product bundles, say $\boldsymbol{U} \times \mathbb{R}^{\boldsymbol{n}}$ and $\boldsymbol{U} \times \mathbb{R}^{\boldsymbol{n}}$. Then locally the map $\mathbf{T}$ sends $(\boldsymbol{u}, \mathrm{v})$ to $(\boldsymbol{u}, \boldsymbol{H}(\boldsymbol{u})\mathrm{v})$, where $\boldsymbol{H}(\boldsymbol{u})$ is a (continuous or smooth) function taking values in the space of all $\boldsymbol{m} \times \boldsymbol{n}$ matrices of rank $\boldsymbol{n}$ (recall that $\boldsymbol{n} < \boldsymbol{m}$). In particular, if as usual we let $\mathbf{e}_1, \ldots$ *etc.* denote the standard unit vectors, then for each $\boldsymbol{u}$ the vectors $\mathbf{a}_i(\boldsymbol{u}) = \boldsymbol{H}(\boldsymbol{u})\mathbf{e}_i$ form a basis for the subspace $\mathbf{T}[\boldsymbol{A_u}]$.

Next, extend the linearly independent set $\{\mathbf{a}_i(\boldsymbol{u})\}$ to a basis for $\mathbb{R}^{\boldsymbol{n}}$ by adding the vectors $\mathbf{w}_j$, where $\boldsymbol{j} = 1, \ldots, \boldsymbol{m} - \boldsymbol{n}$; define cross sections of $\boldsymbol{B}$ restricted to $\boldsymbol{U}$ by $\mathbf{b}_j(\boldsymbol{u}) = (\mathbf{u}, \mathbf{w}_j)$. Since the set of invertible matrices is open in the set of all $\boldsymbol{m} \times \boldsymbol{m}$ matrices, it follows that for all $\boldsymbol{u}$ sufficiently close to $\boldsymbol{x}$ the vectors $\mathbf{a}_i(\boldsymbol{u})$ and $\mathbf{b}_j(\boldsymbol{u})$ combine to form a basis for $\mathbb{R}^{\boldsymbol{n}}$. If necessary, take a smaller neighborhood $\boldsymbol{V}$ of $\boldsymbol{x}$ such that the given vectors form a basis for all points of $\mathbf{V}$.

We can now apply the Gram – Schmidt orthogonalization process to the vector valued functions $\mathbf{a}_1, \ldots, \mathbf{a}_n, \mathbf{b}_1, \ldots, \mathbf{b}_{m-n}$ in that order. Let $\mathbf{y}_1, \ldots, \mathbf{y}_n, \mathbf{z}_1, \ldots, \mathbf{z}_{m-n}$ in $\boldsymbol{B_u}$ be the orthonormal basis obtained in this fashion. Then it follows immediately that for each $\boldsymbol{v}$ in $\boldsymbol{V}$ the vectors $(\boldsymbol{v}, \mathbf{z}_j(\boldsymbol{v}))$ are an orthonormal basis for $\boldsymbol{C_v}$. In particular, using the functions $\mathbf{z}_j$ we obtain a vector bundle inclusion from the product vector bundle $\boldsymbol{V} \times \mathbb{R}^{\boldsymbol{m}-\boldsymbol{n}}$ to $\boldsymbol{B}$, the image of which is the subspace $\boldsymbol{C}$ described above. In the smooth category, this mapping is also a smooth embedding. Thus we know that locally $\boldsymbol{C}$ has a vector bundle structure and maps appropriately into $\boldsymbol{B}$. To conclude the argument we need to show that $\boldsymbol{C}$

has a global vector bundle structure, and this structure is smooth if our original data are smooth.

The preceding discussion gives us an open covering $\{V_\alpha\}$ of X and "chart" homeomorphisms $h_\alpha: V_\alpha \times \mathbb{R}^{m-n} \to C$, $k_\alpha: V_\alpha \times \mathbb{R}^m \to B$, with the following properties:

> If $\mathbf{e}_1, \ldots$ *etc.* denotes the standard basis of unit vectors for $\mathbb{R}^{m-n}$, then for each point p in V_α, the maps h_α sends $(p, \mathbf{e}_j)$ to $k_\alpha(\mathbf{p}, \mathbf{z}_{\alpha,j}(\mathbf{p}))$ where the $\mathbf{z}_{\alpha,j}$ are orthonormal vector valued continuous functions; in the smooth case, the k_α are smooth charts and the $\mathbf{z}_{\alpha,j}$ are smooth functions.

Suppose now that we have an open set Ω contained in both V_α and V_β, so that the restrictions of h_α and h_β both define homeomorphisms from $\Omega \times \mathbb{R}^{m-n}$ to C which differ by a homeomorphism Ψ from $\Omega \times \mathbb{R}^{m-n}$ to itself. What can we say about this map? First of all, since h_α and h_β are fiber preserving maps, the same is true of the difference homeomorphism Ψ. Next, since h_α and h_β are linear maps on each fiber, the same is also true of Ψ. This is enough for us to conclude that we have a topological vector bundle atlas for $C \to X$. We need to show that this will be a smooth atlas if the original data are smooth. Now the $\mathbf{z}_{\alpha,j}$ and $\mathbf{z}_{\beta,j}$ both define orthonormal bases for the same vector subspace when evaluated at an arbitrary point p of Ω. Therefore the standard formulas of linear algebra imply the existence of an $(m-n) \times (m-n)$ matrix valued function $G_{\beta,\alpha}(p)$ such that the entries of $G_{\beta,\alpha}(p)$ are smooth functions of p (in fact, they are inner products of $\mathbf{z}_\alpha$ and $\mathbf{z}_\beta$ vectors) and the matrix $G_{\beta,\alpha}(p)$ sends $\mathbf{z}_{\alpha,j}(p)$ to $\mathbf{z}_{\beta,j}(p)$ for all j and p. This means that the map Ψ has the form $\Psi(p, \mathbf{w}) = (p, G_{\beta,\alpha}(p)\,\mathbf{w})$ and hence is a diffeomorphism which sends fibers to fibers linearly. Therefore the mappings h_α define a smooth vector bundle atlas for C, which is what we wanted to prove.

In particular, this means that the objects denoted by Ξ (upper case Greek xi) on pages 93 – 94 and 99 – 100 of Bredon are smooth vector bundles and hence also smooth manifolds. In this language, the arguments in Bredon show that the smooth maps in the proof of the Tubular Neighborhood Theorems have invertible derivatives on the zero sections of the given vector bundles, and therefore they are locally diffeomorphisms at such points by the Inverse Function Theorem. Once we have this, the rest of the arguments in Bredon go through without change.

An application of the Tubular Neighborhood Theorem

In his book, ***Topology and Geometry***, Bredon uses the Tubular Neighborhood Theorem to show that continuous maps on smooth manifolds can be approximated by smooth ones. We shall mention another result in this direction which is implicit in Bredon and very important for the standard applications of algebraic topology to smooth manifolds:

Proposition. Let N^n be a smooth submanifold M^m (where $n < m$). Then there is an open neighborhood U of N in M such that N is a strong deformation retract of M.

Proof. Take U to be a tubular neighborhood of N. Let $q: U \rightarrow N$ correspond to the vector bundle projection, so that the inclusion j of N in U corresponds to the zero section. Then the composite $q\,j$ is the identity on N, and the reverse composite $j\,q$ is homotopic to the identity by the elementary straight line homotopy $H_t(\mathbf{v}) = t\,\mathbf{v}$.