Vector bundles over cylinders

One important fact about vector bundles is that they can be classified using homotopy theory. An essential step in developing this classification is the following result on vector bundles over the product of a space with an interval:

Theorem. Let $\alpha: A \to X$ and $\beta: B \to X$ be (topological or smooth) vector bundles over a **compact** space X, let

$$\alpha \times \textbf{Identity}: A \times (-\varepsilon, 1+\varepsilon) \to X \times (-\varepsilon, 1+\varepsilon)$$

denote the product with the open interval $(-\varepsilon, 1+\varepsilon)$ for some $\varepsilon > 0$ and likewise for $\beta \times$ **Identity,** for each x in X let A_x and B_x denote the vector spaces (fibers) in A and B which map to x under the projection maps, and let

$$\mathbf{T}: A \times (-\varepsilon, 1+\varepsilon) \longrightarrow B \times (-\varepsilon, 1+\varepsilon)$$

be a (continuous or smooth) mapping such that for each x in X and t in the interval, the mapping $\mathbf{T}$ sends $A_x \times \{t\}$ to $B_x \times \{t\}$ by a linear monomorphism. Assume further that we are given a (continuous or smooth) Riemannian metric on B, let C denote the orthogonal complement of $\mathbf{T}[A \times \{0\}]$ in $B \times \{0\}$ under the obvious identifications $Y \to Y \times \{0\}$, and let D denote the orthogonal complement of $\mathbf{T}[A \times (-\varepsilon, 1+\varepsilon)]$. Then D is isomorphic as a (continuous or smooth) to the product $C \times (-\varepsilon, 1+\varepsilon) \to X \times (-\varepsilon, 1+\varepsilon)$.

It is possible to eliminate the compactness assumption on X, but we shall not do so since we only need the result in the compact case. In the topological category there is a similar result if one replaces the open interval with a closed interval such as $[0, 1]$; the argument is very similar to the one given below, and it is left to the reader as an exercise.

Important consequences. **1.** ***Isotopic smooth embeddings have isomorphic normal bundles.*** Given a smooth isotopy $h_t: M \to N$, there is an associated smooth embedding

$$k: M \times (-\varepsilon, 1+\varepsilon) \longrightarrow N \times (-\varepsilon, 1+\varepsilon)$$

sending (x, t) to $(h_t(x), t)$. The preceding theorem implies that the normal bundle to the embedding k is a product of the normal bundle to $N \times \{0\}$ with the interval, and the restriction of this normal bundle to $N \times \{1\}$ is the restriction of the normal bundle for k to $N \times \{1\}$.

2. If we use the modified version of the main result for topological products with closed intervals, by induction we can conclude that every topological vector bundle over a product of closed intervals — and hence also every topological vector bundle over a closed disk — is a product bundle.

3. Using the preceding, we can conclude that every k – dimensional vector bundle over the sphere S^n is obtained by gluing together two k – dimensional product bundles over the upper and lower hemispheres by means of a vector bundle automorphism over the equatorial subsphere S^{n-1}. Such an automorphism corresponds to a continuous map from the subsphere into the group $GL(k, \mathbb{R})$ (this map is often called a ***clutching function***); one can prove that two clutching functions determine the same vector bundle if and only if they are homotopic (we shall prove variants of this result later). Similar results hold in the smooth category but require more work.

Proof of the theorem. There are two main steps in the proof. The first is to show that for each choice of t there is an $h > 0$ such that the restriction of D to the thickened slice $X \times (t-h, t+h)$ is a product of $C[t]$ — the restriction of the vector bundle D to $B \times \{t\}$ — with the interval $(t-h, t+h)$. The second step of the proof applies the first to conclude that D itself is isomorphic to the product $C \times (-\varepsilon, 1+\varepsilon)$.

Let $E(t)$ denote the fiberwise perpendicular projection of $B \times \{t\}$ onto $C[t]$, and let x be a point in X. Then a continuity argument (which can be done over an open subset of X on which B is trivial) implies that over all points (y, u) sufficiently close to (x, t) the restriction of $E(t)_x$ to $C_y \times \{u\}$ maps the latter isomorphically to $C_x \times \{t\}$. In particular, it follows that there is an open neighbhorhood W of $X \times \{t\}$ such that the restriction of the composite

$$D \subset B \times (-\varepsilon, 1+\varepsilon) \to B \times \{t\} \to C[t]$$

to the inverse image of W is an isomorphism on each fiber. Since X is compact, it follows that there is some $h > 0$ such that $X \times (t-h, t+h)$ is contained in W. But this implies that the restriction of the bundle D to $B \times (-\varepsilon, 1+\varepsilon)$ is isomorphic to $C[t] \times (t-h, t+h)$.

In particular, for each value of t it follows that there is some $h > 0$ (depending upon t) such that the restriction of D to $X \times (t-h, t+h)$ is isomorphic to the product $C[t] \times (t-h, t+h)$; let h_0 be the choice of h corresponding to $t = 0$. Consider now the set of all $k > 0$ such that the restriction of D to $X \times (-h_0, k)$

is isomorphic to the product of $\boldsymbol{C}[\mathbf{0}]$ with $(-h_0, k)$; this nonempty set is bounded from above by $\mathbf{1} + \varepsilon$ and hence has a least upper bound η. We claim that η is equal to $\mathbf{1} + \varepsilon$. If it is strictly less than $\mathbf{1} + \varepsilon$, consider what must happen. First, since η is the least upper bound it follows that for all $k < \eta$ the restriction of D to $X \times (-h_0, k)$ is isomorphic to $C[0] \times (-h_0, k)$. Next, there is some $\delta > 0$ such that the portion of the bundle over $(\eta - \delta, \eta + \delta)$ is a product of $C[\eta]$ with $(\eta - \delta, \eta + \delta)$. Now we know that the restriction of D to $X \times (-h_0, \eta - [\delta/3])$ is isomorphic to $C[0] \times (-h_0, \eta - [\delta/3])$. If we combine these observations, one immediate conclusion is that $C[0]$, $C[\eta - \frac{1}{2}\delta]$ and $C[\eta]$ are all isomorphic. Using these isomorphisms, we can piece together an isomorphism between the restriction of D to $X \times (-h_0, \eta + \delta)$ and $C[0] \times (-h_0, \eta + \delta)$. This contradicts our basic assumption that the least upper bound η was strictly less than $1 + \varepsilon$, and therefore we must have $\eta = 1 + \varepsilon$. This almost gives the desired conclusion, the difference being that we only know the result for the restriction to the subset $X \times (-h_0, 1 + \varepsilon)$. However, we can now modify the preceding argument to show that D, which is its restriction to $X \times (-\varepsilon, 1 + \varepsilon)$, is isomorphic to the product $C[0] \times (-\varepsilon, 1 + \varepsilon)$.