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# Topology Seminar, 2/2/14

**Last time** Proved the following:

**Theorem** $X$ compact, connected, metrizable ($\Longleftrightarrow$ $X$ connected, compact $T_2$, $2^{nd}$ countable).

Then $X \cong [0,1] \Longleftrightarrow$

① there are exactly two points $p, q$ s.t. $X - \{p\}$, $X - \{q\}$ is connected.

② for all other points, $X - \{x\}$ has two connected components, each of which meets $\{p, q\}$ in one point.

Actually the last part (each of which...) is redundant, depending upon more extensive results concerning compact, connected metrizable spaces (continuum theory). Two references

Hocking & Young, Topology.

Willard, General Topology.

Want similar characterizations of $S^1$ and $\mathbb{R}$.

The result for $S^1$ is in Hocking and Young

<u>Thm.</u> $X \cong S^1 \Leftrightarrow X$ compact, connected, metrizable and

(1) Each set $X - \{x\}$ is connected.

(2) for $x \neq y$ in $X$, the set $X - \{x, y\}$ has two components.

This also requires some lengthy digressions into continuum theory, so we shall only prove a weaker result:

<u>Theorem (weak version)</u> $X$ compact, connected, metrizable. Then $X \cong S^1 \Leftrightarrow$

(1) each $X - \{x\}$ is connected

(2) for $x \neq y$ in $X$, $X - \{x, y\}$ has two components $U + V$ s.t. $\mathrm{Bdy}\, U = \mathrm{Bdy}\, V = \{x, y\}$

<u>AND</u>

③ for distinct points $x, y, z \in X$, $X - \{x, y, z\}$ has three components, and their boundaries are $\{x,y\}$, $\{y,z\}$, $\{x,z\}$ respectively.

**Proof of the weak result**

Write $X - \{x, y, z\} = U \cup V \cup W$ pairwise disjt, open connected

with $\operatorname{Bdy} U = \{y, z\}$, $\operatorname{Bdy} V = \{x, z\}$, $\operatorname{Bdy} W = \{x, y\}$.

Then $X - \overline{W} = U \cup \{z\} \cup V$ and

$$X - W = \{y\} \cup U \cup \{z\} \cup V \cup \{x\}.$$

Let's examine $X - W \overset{=F}{}$ more closely. Its boundary is $\{x, y\}$ (notice $z \in \overline{U} \cap \overline{V}$).

It follows that
$$\{y\} \cup U \cup \{z\} \cup V = F - \{x\}$$
$$U \cup \{z\} \cup V \cup \{x\} = F - \{y\}$$
are connected open subsets of $F$, so $x, y \in F$ are not cut points. CLAIM every other point is a cut point.

Suppose that $t \in F - \{x, y\}$. Then $X - \{x, y, t\}$ has 3 components, etc., and one of them must be $W \subseteq X - \{x, y, t\}$. because $W$ is a connected component of the larger set $X - \{x, y\}$. By the assumptions, $X - \{x, y, t\}$ $= U_t \cup V_t$ where $U_t$ and $V_t$ are open, connected and disjoint. and $t \in L(U_t) \cap L(V_t)$ limit point sets.

In particular, we also have that

$$\mathrm{Bdy}\ U_t = \{x, t\}, \quad \mathrm{Bdy}\ V_t = \{y, t\}$$

or vice versa. SAY THE DISPLAYED EQUATIONS HOLD.

Then $F - \{t\}$ is the union of the open connected subsets $U_t \cup \{x\}$, $V_t \cup \{y\}$. In fact

these sets are closed in $F - \{t\}$ since

$$\text{Closure } \begin{matrix} U_t \\ V_t \end{matrix} \text{ in } X = \begin{matrix} U_t \cup \{x, t\} \\ V_t \cup \{y, t\} \end{matrix}.$$

This means that $F \cong [0,1]$ by the previous result, for

$$\left.\begin{array}{l} V_t \cup \{x\} \\ V_t \cup \{y\} \end{array}\right. \quad \text{are open closed}$$

connected subsets of T which are disjoint and whose union is $F - \{t\}$.

Similar considerations show that if $\Omega$ is the other component of $X - \{x,y\}$ and $E$ is the closed set $\Omega \cup \{x,y\}$, then $E$ has $x$ and $y$ as noncut points and $t \in \Omega \Rightarrow E - \{t\}$ has two components, each of which contains one point of $\{x,y\}$.

Hence we also have $E \cong [0,1]$, so that

$$X \cong [0,1] \underset{\substack{\text{disjt} \\ \text{union}}}{\coprod} [0,1] \quad \text{identify zeros and ones.}$$

But this means $X \cong S^1$. $\square$

How to characterize $\mathbb{R} \cong (0,1)$?

<u>Thm. (A.J. Ward, Quart. J. Math Oxford (2), 41 (1936), 191-198).</u> $X \cong \mathbb{R} \Leftrightarrow X$ is

$\boxed{T_3 \Leftrightarrow}$ metrizable, second countable, locally connected, connected, and for each $x \in X$, $X - \{x\}$ has two open connected (noncompact) components.

<u>Idea</u>: Use connectedness to define a partial ordering, analyze it somewhat like the case of characterizing $\mathbb{R}$.

Here is a simple approach using the characterization of $[0,1]$.

<u>Alternate characterization.</u> $X \cong \mathbb{R} \Leftrightarrow$
$X$ is locally compact $T_2$, second countable, connected, and

① For each $x \in X$, $X - \{x\}$ has two connected open components $U$ & $V$ with $\overline{U}, \overline{V}$ noncompact,

(2) for $x \neq y$ in $X$, the set $X - \{x, y\}$ has 3 connected components $U, V, W$ such that

(a) $U$ is a component of $X - \{x\}$

(b) $W$ is a component of $X - \{y\}$

(c) $\overline{V}$ is compact.

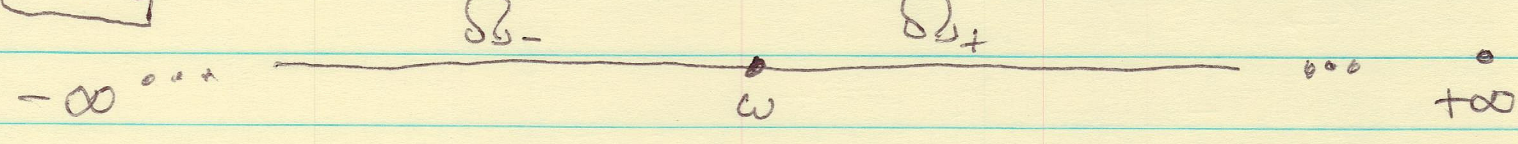
<u>Proof</u> First step:

$\operatorname{Bdy} U = \{x\}$, $\operatorname{Bd} W = \{y\}$, $\operatorname{Bdy} V = \{x, y\}$, where $x \neq y$ are arbitrary.

Compactification $\omega \in X$, $X - \{\omega\} = \Omega_+ \cup \Omega_-$.

$\overline{\Omega_+} = \Omega_+ \cup \{\omega\}$, $\overline{\Omega_-} = \Omega \cup \{\omega\}$.

$\boxed{C_\pm \text{ loc. comp. } T_2}$ Let $C_\pm$ = one pt. compactification of $\overline{\Omega_\pm}$.




$$Y = C_+ \cup_{\{\omega\}} C_-.$$

Need to show $Y$ has non cut points $\pm\infty$, $Y - \{x\}$ has two connected components, $Y$ compact $T_2$ second countable.

Notice that $C_\pm$ and $C_\pm - \{\infty\}$ are connected, and so is $Y$.

Also $Y$ is compact and $T_2$ (standard exercises $Y = C_+ \cup C_-$ where $C_\pm$ compact $\overset{T_2}{\Rightarrow}$ $Y$ is also compact $T_2$).

Slightly less trivial: $Y$ is second countable. Need to show $C_\pm$ has a countable nbhd base at $\infty$. In other words $C_\pm$ is an increasing union $\cup K_{n\pm}$ where $K_{n\pm} \subseteq K_{n+1\pm}$ compact.

This follows from local compactness and second countability ($C_\pm$ is a countable union of open sets with compact closures).

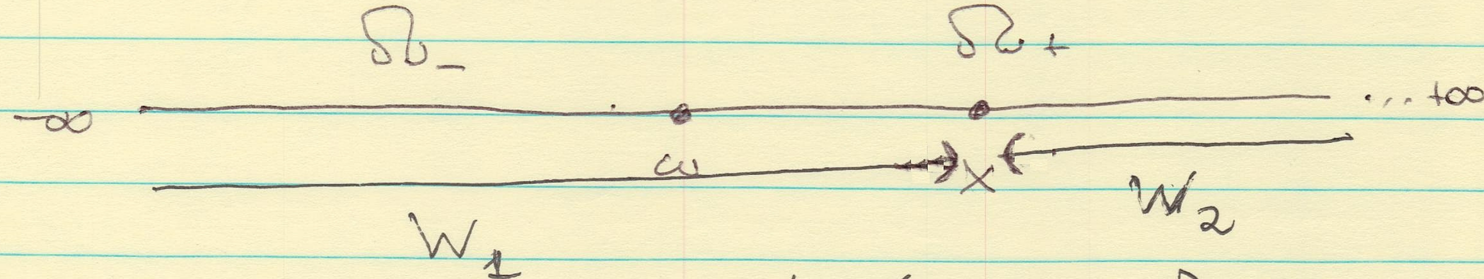
Need to check the cut point assertions.

<u>noncut</u>: $Y - \{-\infty\} = \overline{\Omega_-} \cup C_+$,
$Y - \{+\infty\} = \overline{\Omega_+} \cup C_-$, both connected.

$\boxed{\text{Now look at } Y - \{x\}}$ $\quad \Omega_+ \cup \{+\infty\} \quad \Omega_- \cup \{-\infty\}$

$\underline{x = \omega}$ two components $C_+ - \{\omega\}$ and $C_+ - \{\omega\}$

$\underline{x \neq \omega}$ Without loss of generality, assume $x \in \Omega_+$ (other case is similar)




$\infty \in W_1$ (W.L.O.G.)

$$X - \{\omega, x\} = \Omega_- \cup V \cup W_2$$

$\Omega_- \cup \{-\infty\}$ is open connected $= C_- - \{\omega\}$.

To conclude, we need to show that

$W_2 \cup \{+\infty\}$ is open and connected.

$\overline{V} = V \cup \{\omega, x\}$

openness: $W_2 \cup \{+\infty\} = Y - (C_- \cup \overline{V})$

$C_-$ compact, $\overline{V}$ compact $\Longrightarrow$ union closed!

To see that $W_2 \cup \{+\infty\}$ is connected

Two steps

(A) $S$ is locally compact $T_2$, $S^* =$ one pt. compactification, $K \subseteq S$ compact, $W = S - K$. Then $\infty$ is a limit point of $W$. [Let $V \cup \{\infty\}$ be an open nbhd of $\infty$, so $S - V$ is a compact set $L$. Then $V \cap W \neq \phi$ because $V \cap W = \phi \Rightarrow S = K \cup L$, which is compact.]

(B) By the preceding, $\infty$ is a limit pt. of $W_2$. Since $W_2$ is connected, so is $W_2 \cup \{\infty\}$.

Note

$W_2 =$

$\overline{\Omega_+} - \overline{V}$.

$\Rightarrow W_2$ open in $\overline{\Omega_+}$.