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Problem Find a set of "axioms" to characterize the closed interval $[0,1]$.

Theorem. X compact, connected, metrizable topological space. Then $X \cong [0,1] \Longleftrightarrow$

① there are exactly two points p, q s.t. $X - \{p\}$ & $X - \{q\}$ are connected.

② for all other points, $X - \{x\}$ has two components, each of which meets $\{p, q\}$ in a single point.

Idea Define a linear ordering on X with p = minimum elt, q = maximum elt.

Show the order topology equals the given topology, and the space is homeo to $[0,1]$.

Ordering u, v neither is p or q.

Say $u \leq v$ if the connected component of $X - \{u\}$ containing p is a subset of the connected component of $X - \{v\}$ cont. p.

CLAIM This is a linear ordering.

Reflexive — Trivial. Transitive — Same

Trichotomy — Say $u \neq v$; then either v is in the component of $X - \{u\}$ containing p or v is in the component of $X - \{u\}$ cont. q. In the first case, $v < u$. Suppose that the second alternative holds; let L_v + U_v be the components of $X - \{v\}$ containing p + q respectively, and likewise for L_u + U_u. Then our hypothes translates to $v \in U_u$; we want to show $u \in L_v$. But $u \in U_v \Rightarrow$ the connected set L_u must be contained in the component U_v of $X - \{v\}$ since $v \notin L_u$. This implies

$p \in L_u \subseteq U_v$. But $p \notin U_v$ by the hypotheses, so we have a contradiction. Therefore we must have $u \in L_v$, so that $u < v$. ∎

Note that L_u & U_u are open in $X - \{w\}$ $\Rightarrow$ they are open in X.

CLAIM Order topology = given topology. Let $J : (X, T) \longrightarrow (X, \leq)$ be the identity. Since X is compact Hausdorff & $(X, \leq)$ is Hausdorff, it suffices to show J is continuous. What are basic open subsets in $(X, \leq)$?

$$U_v, \; L_v : v \in X - \{p, q\}$$
$$X - \{p\}, \; X - \{q\}.$$

In each case, the sets are open in the original topology T on X, so J is cont. Since it is 1-1, onto, it must be a homeo ∎

For the next step, we use the fact that X is second countable. Let D be a countable dense subset.

Note that if $u < v$ in X, then $\exists\, d \in D$ s.t. $u < d < v$.

<u>PROOF</u> Need only show the open interval (u, v) is nonempty. Suppose not. Then
$p \leq u < v \leq q \Rightarrow X$ is the union of the disjoint closed subsets $[p, u]$ & $[v, q]$, which means X is disconnected.

We might as well assume $p, q \in D$ and take a 1-1 correspondence $D \longleftrightarrow \mathbb{N}$ s.t. $p = d_0$, $q = d_1$.

Define a 1-1 order-preserving correspondence $D \longleftrightarrow$ finite binary fractions

$$a_0 + \frac{a_1}{2} + \cdots + \frac{a_k}{2^k} + \cdots + \frac{a_N}{2^N} \text{ (finite)!}$$

$[$if $a_0 = 1$ then $a_i = 0$ for $i \geq 1]$

$d_0 \xrightarrow{f} 0$ $\quad d_1 \xrightarrow{f} 1$. Suppose $d_1, \ldots, d_k$ defined, $k \geq 1$. Consider d_{k+1}. $\exists\, d_i$ & d_j s.t.

$d_i < d_{k+1} < d_j$ but $l \leq k$ then

$d_i < d_l < d_j$ is FALSE.

(up to iso, only one lin. ordered set w/ (k+1) elts.)

Set $f(d_{k+1}) = \frac{1}{2}\left(f(d_i) + f(d_j)\right)$.

By construction f is 1-1. Claim f is onto

① $u, v \in$ Image $f \Rightarrow$ so is $\frac{u+v}{2}$, for $u = f(d_i)$ + $v = f(d_j) \Rightarrow (u,v) \cap D \neq \emptyset$ $\Rightarrow$ for some k, $f(d_{k+1}) = \frac{1}{2}(f(u) + f(v))$.

② $\{0, 1\} \subseteq$ Image f. $\Rightarrow \{0, \frac{1}{2}, 1\} \subseteq$ Im f $\Rightarrow \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\} \subseteq$ Im f etc...

Let g: Finite binary fractions $\xrightarrow{\cong} D \subseteq X$ inverse to f; g is strictly $\uparrow$.

Now extend f to all of X as follows:

$$x \neq p, q \Rightarrow F(x) = \underset{\substack{d \in L_x \\ d \in D}}{\text{L.U.B.}} f(d).$$

$d \in L_x \Leftrightarrow [d \in D + d < x]$

CLAIM: ① $x \neq p, q \Rightarrow 0 < F(x) < 1$.

② F is strictly increasing.

③ $F(d) = f(d)$ if $d \in D$.

④ F is onto

PROOFS ② Say $x < y$ $(< q)$. Then $\exists\, d^*$ s.t.
$x < d^* < y$, so $\{f(d) \mid d < x\} \neq \{f(d) \mid d < y\}$ since f is 1-1

But $\{f(d) \mid d < x\} \subseteq \{f(d) \mid d < y\}$, so we must have proper containment and hence
$F(x) < F(y)$ ■

① If $x < q$, then $x < y < q$ for some y,
so $F(x) < F(y) \leq 1$ ■

③ Need to show $f(d) = \mathrm{L.U.B.}\{f(e) \mid e < d, e \in D\}$

Since f is strictly increasing $f(d) > f(e)$ if $e \in D$ + $e < d$, so $f(d) \geq F(d)$. If $f(d) > F(d)$ then $\exists\, e \in D$ s.t. $f(d) > f(e) > F(d)$, and we have $d > e$. By definition we would have $F(d) \geq f(e)$, a contradiction. Hence $f(d) \leq F(d)$ + we must have $f(d) = F(d)$. ■

④ Take a sequence of intervals $[a_k, b_k]$ s.t. $[a_k, b_k] \supseteq [a_{k+1}, b_{k+1}]$
$a_k, b_k \in D^*$, $\cap [a_k, b_k] = \{t\}$.
$(0 < t < 1)$. $D^* \subseteq [0,1]$ binary fractions.

The inverse image E_k of $[a_k, b_k]$ in X is a closed interval because $a_k, b_k \in F[D]$.

Since $E_k \supseteq E_{k+1}$, by compactness we must have $\cap E_k \neq \emptyset$, and if $x \in \cap E_k$ then $F(x) = t$ because of ② + ③. ■

<u>CONCLUSION</u> F is an order preserving 1-1 correspondence $X \longrightarrow [0,1]$. Since $(X, T) = (X, \leq)$ this means that F must be a homeomorphism. ■