Math 510 – Schultz

Fall 1995

Applications to the Heat Equation

The purpose of this section is to explain how the Divergence Theorem can be used to derive the partial differential equation for heat conduction. A substantial portion of this material is adapted from the book, *Introduction to Partial Differential Equations with Applications*, by E. C. Zachmanoglou and D. W. Thoe, especially page 156.

In dealing with applications of mathematics it is often necessary to develop some theoretical material that may be of independent interest. We shall begin with some items that will be needed in the derivation of the Heat Equation.

Preliminaries

The first thing we need is a mean value theorem for multiple integrals.

Mean Value Theorem. *Suppose that the function f is continuous on a Type* I *region D defined by inequalities of the form $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, $h_1(x,y) \leq z \leq h_2(x,y)$, where the preceding functions are continuous at the appropriate points. Then there is a point (X, Y, Z) in D for which*

$$\int\int\int_D f(x,y,z)\,dx\,dy\,dz = f(X,Y,Z) \cdot \text{Volume}(D).$$

The description of Type I regions in the theorem corresponds to the usage in the book, *Vector Calculus*, by J. Marsden and A. Tromba; actually the definition in the latter also allows similar sets of inequalities where the roles of x and y are reversed.

Sketch of derivation. Basic theoretical results on continuous functions imply that f takes a minimum value m and a maximum value M on the set D. By the properties of integrals on pages 319 of Marsden and Tromba we have

$$m \cdot \text{Volume}(D) \leq \int\int\int_D f(x,y,z)\,dx\,dy\,dz \leq M \cdot \text{Volume}(D)$$

and if we divide these inequalities by the volume of D we obtain the inequalities

$$m \leq \frac{\int\int\int_D f(x,y,z)\,dx\,dy\,dz}{\text{Volume}(D)} \leq M.$$

Another basic fact about continuous functions on connected sets like D states that f takes every value between its minimum and maximum. Choose (X, Y, Z) so that $f(X, Y, Z)$ takes the value of the middle term in the second string of inequalities; if both sides of the associated equation are multiplied by the volume of D, the equation in the theorem will be obtained.■

This leads to another useful result that is sometimes called the *Differentiation Theorem* for multiple integrals.

Theorem. *Let $f(x, y, z)$ be continuous on an open set U containing $\mathbf{v}_0 = (x_0, y_0, z_0)$ and for all $r > 0$ let $D(r)$ be the closed disk defined by $|\mathbf{v} - \mathbf{v}_0| < r$. Then we have*

$$f(\mathbf{v}_0) = \lim_{r \to 0} \frac{1}{\text{Volume}(D(r))} \int\int\int_{D(r)} f(x, y, z)\, dx\, dy\, dz.$$

Sketch of derivation. According to the preceding Mean Value Theorem the expression on the right hand side is equal to $f(X_r, Y_r, Z_r)$ for some point (X_r, Y_r, Z_r) in $D(r)$. Since the points in question satisfy

$$\lim_{r \to 0} (X_r, Y_r, Z_r) = (x_0, y_0, z_0)$$

by continuity we also have

$$\lim_{r \to 0} f(X_r, Y_r, Z_r) = f(x_0, y_0, z_0)$$

and if we make the substitution

$$f(X_r, Y_r, Z_r) = \frac{1}{\text{Volume}(D(r))} \cdot \int\int\int_{D(r)} f(x, y, z)\, dx\, dy\, dz$$

we obtain the formula in the theorem.■

We need one more piece of mathematical input.

Differentiation under the integral sign. *Suppose that $F(s, t)$ is a function of several variables with continuous partial derivatives for appropriate values of s and t, and let*

$$f(t) = \int_D F(s, t)\, ds$$

(the integral can be an ordinary or multiple integral depending on the number of variables represented by s). Then f has a continuous derivative that is given by

$$f'(t) = \int_D \frac{\partial F(s, t)}{\partial t}\, ds.$$

The derivation of this formula is not difficult and follows from the principles discussed in Section 5.5 of Marsden and Tromba; details can be found in many undergraduate texts on functions of real variables.■

Derivation of the Heat Equation

Let U be a region in $\mathbf{R}^3$, and let $u(x, y, z, t)$ be a function corresponding to the temperature distribution as a function of the space coordinates $(x, y, z) \in U$ and a real time coordinate t.

Given a point $(x, y, z) \in U$, let D be a small closed disk containing (x, y, z) that lies entirely in U, and let S be its boundary sphere. The outward normal at a point of S will be denoted by $\mathbf{n}$. If $q(t)$ denotes the thermal energy in D at a given time t, the following physical law relates the thermal energy function q to the temperature distribution function u:

$$-\frac{dq}{dt} = -\int\int_S k(x, y, z)\frac{\partial u}{\partial \mathbf{n}}\, dS = -\int\int_S k(x, y, z)(\nabla u) \cdot d\mathbf{S}$$

where k is the thermal conductivity of the material filling the region U and ∇ is taken with respect to x, y, z. we are assuming that the material is ISOTROPIC; in other words, k does not depend on the normal direction to S but only on the x, y, z coordinates.

Consider the change in q from time t to time $t + \Delta t$:

$$\Delta q = \int\int\int_D c(x, y, z)\rho(x, y, z)[u(x, y, z, t + \Delta t) - u(x, y, z, t)]\, dx\, dy\, dz$$

where c represents the specific heat of the material and ρ represents the density.

If we form the difference quotient

$$\frac{\Delta q}{\Delta t}$$

take limits as $\Delta t \to 0$, and apply differentiation under the integral sign we obtain the equation

$$\frac{dq}{dt} = \int\int\int_D c(x, y, z)\rho(x, y, z)\frac{\partial}{\partial t}u(x, y, z, t)\, dx\, dy\, dz.$$

On the other hand, the previously mentioned physical law and the Divergence Theorem combine to imply that

$$\frac{dq}{dt} = \int\int\int_D \nabla \cdot (k\nabla u)\, dx\, dy\, dz$$

and standard identities involving ∇ imply that the integrand in the latter expression is equal to $\nabla k \cdot \nabla u + k\nabla^2 u$. Therefore we have shown that the triple integrals of the functions

$$c\rho\frac{\partial u}{\partial t} \qquad \text{and} \qquad \nabla k \cdot \nabla u + k\nabla^2 u$$

are equal over the disk D, and therefore the quotients of these integrals by the volume of D are also equal. Since the radius r of D is arbitrary, the limits of these expressions as $r \to 0$ are equal to the integrands. Thus the equality of the triple integrals implies equality of the integrands; more precisely, we have

$$c\rho\frac{\partial u}{\partial t} = \nabla k \cdot \nabla u + k\nabla^2 u.$$

If we assume that c, ρ and k are constant this reduces to the standard *Heat Equation*

$$\frac{c\rho}{k}\frac{\partial u}{\partial t} = \nabla^2 u.$$

Steady state temperature distributions. Physical considerations suggest that a temperature distribution will stabilize to some equilibrium distribution after a while. Mathematically, this means that we expect the function u to satisfy a relationship of the form $\lim_{t\to\infty} u(x,y,z,t) = w(x,y,z)$ where $w(x,y,z)$ represents the equilibrium temperature distribution. This well known physical phenomenon is also predicted theoretically by the theory of partial differential equations and boundary problems, but such topics are beyond the scope of this course. A little formal manipulation suggests that the equilibrium temperature satisfies the *Laplace equation* $\nabla^2 w = 0$.

Here is a STANDARD EXERCISE motivated by the preceding discussion: *Use Green's First Identity to show that if $\nabla^2 u = 0$ on a reasonable closed region R and $u = 0$ on the boundary Σ of R, then $u = 0$ on all of R.*

This has the following physical meaning: If R represents a homogeneous, isotropic body and u is the equilibrium temperature distribution with $u = 0$ on the boundary S, then $u = 0$ everywhere.— Of course this corresponds to physical experience, so from a physical viewpoint the significance is that it provides evidence to corroborate the standard theory of heat in physics.

Derivation. As in Marsden and Tromba, Green's First Identity is given by

$$\int\!\!\int\!\!\int_R u\nabla^2 w\,dx\,dy\,dz = \int\!\!\int_S u\frac{\partial u}{\partial \mathbf{n}}\,d\sigma - \int\!\!\int\!\!\int_R \nabla u\cdot\nabla w\,dx\,dy\,dz.$$

If we set $u = w$ and recall that $\nabla^2 u = 0$ then we obtain

$$0 = \int\!\!\int_S u\frac{\partial u}{\partial \mathbf{n}}\,d\sigma - \int\!\!\int\!\!\int_R |\nabla u|^2\,dx\,dy\,dz.$$

Since $u = 0$ on the boundary the first integral vanishes, so therefore the integral of $|\nabla u|^2$ is zero; but this integrand is a continuous and nonnegative function, and therefore the vanishing of the integral implies that the function $|\nabla u|^2$ is identically zero. But the latter in turn implies that $\nabla u = 0$, which means that u is constant. Since $u = 0$ on the boundary, it follows that $u = 0$ everywhere.■

Corrected version, April 1996.