Example of preparation of new material

Here is a sample preparation involving the inverse sine function. The relevant portion of our current Mathematics 5 text (Dugopolski, Third Edition) is contained in Section 5.5.

Checking the textbook. On page 444 of the textbook the author defines the inverse sine function rapidly using the concept of inverse function. Although the students have already seen one concrete example of this abstract concept in exponential and logarithmic functions, it is likely that many if not most students would be better served a more specific discussion of the construction of the inverse sine function in class before saying anything explicitly about defining it as an inverse function.

Starting with concrete motivation. Recall that as x increases from $-\pi/2$ to $+\pi/2$ the function $\sin x$ increases from -1 to $+1$, and that different values of x yield different values of $\sin x$.

It is very useful to draw the graph of $f(x) = \sin x$ *in order to illustrate this*.

At this point the crucial point can be stated: We can in fact go backwards. If u is any number between -1 and $+1$ then there is one and only one value of v between $-\pi/2$ and $+\pi/2$ such that $u = \sin v$.

There are several ways to convince someone that this equation has a solution. One way is to draw the graphs of $y = \sin x$ and $y = u$ in the coordinate plane (*Draw this*.). The picture indicates that the two graphs meet at a single point (v, u) for which v lies between $-\pi/2$ and $+\pi/2$. This can also be seen by geometric experimentation as follows. Suppose that the real number x is strictly between 0 and 1. On a large sheet of paper mark off a line segment AB that is $12\,x$ inches long (in other words its length is x feet). Draw a perpendicular BC to AB at B. Take a ruler that is one foot long, hold one end at A and slide it on the paper so that the other end lies on the ray starting at B and going through C as in the picture below. The ruler will meet the ray at some point D, and if θ is the measure of angle ADB, then we have $x = \sin\theta$. If you have a physical prop that can serve as a substitute for a ruler it might be good to illustrate this process physically at the board.
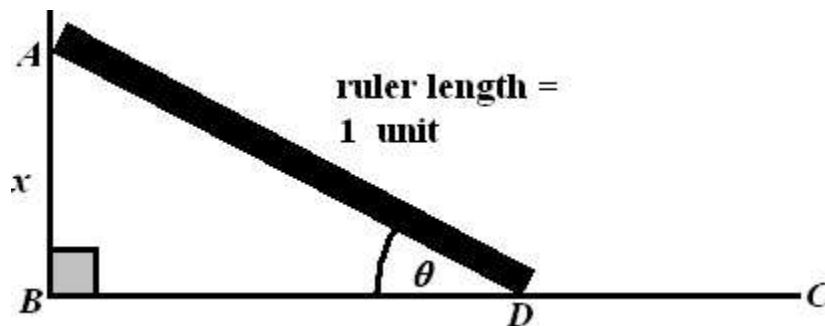
Solving the equation $x = \sin\theta$ by geometric experimentation.

This might be a good point to answer the question, "What happens if x is negative?" One way of addressing such a question is to remind students that $\sin(-v) = -\sin v$, so if u is negative we can find the positive value w so that $\sin w = |u|$ and then notice that $v = -w$ will be the solution that we want.

Provide some specific examples. Remind the students that the solutions to this equation have already been determined for certain values like 0, 1, ½ or half the square roots of 2 or 3. But what if x is some other value like ¾? Exactly what value of θ does one obtain in this case? One approach to finding the solution would be to go through the preceding construction, draw the triangle and measure the angle with a protractor. Another way would be to look at a table of trigonometric functions and use this to find where the value of the sine function is close to ¾. Either method can be used to show that the degree value equivalent to θ is somewhere between 48.5 and 48.6 degrees.

State the first version of the definition. One can now define the inverse sine function for x between -1 and $+1$ by saying that $\sin^{-1}x$ is the unique value of y between $-\pi/2$ and $+\pi/2$ such that $x = \sin y$. In other words we have

$$y = \sin^{-1}(x) \quad \textit{if and only if} \quad x = \sin y$$

Relate to the general concept of inverse function. One can now point out that the definition of $\sin^{-1}x$ is a special case of an inverse function: When f is a function such that $f(x)$ increases from c to d as x increases from a to b (with different values for f at different points), then the *inverse function* f^{-1} is defined for x between c and d and satisfies the fundamental identity

$$y = f^{-1}(x) \quad \textit{if and only if} \quad x = f(y)$$

It is worthwhile to remind the students that this is the second example of an inverse

function that has been encountered in the course and to add that the motivation for introducing abstract concepts like inverse functions is that they arise repeatedly in different contexts. One can also mention that unified approaches of this sort turn out to be especially useful in calculus.

Anticipate issues that will cause difficulties for students.   It is almost certain that some students will look at the previous equation and ask questions like, "Since $\sin \pi = 0$, why can't we also say that $\pi = \sin^{-1} 0$?" One stock answer is that that we have to make a choice because a function is supposed to take only one value for each choice of x. Beyond this one can say, "Experience shows that choosing the values between $-\pi/2$ and $+\pi/2$ is the most convenient option for most purposes."

Another problem is that the *notation for inverse trigonometric functions varies somewhat from one book to the other*, and it is possible that students will see different notation in textbooks for courses in other subjects. Therefore, it may be worthwhile to mention other standard ways of writing the inverse sine function like arcsin x or Arc sin x (the latter often being used to stress that the value of the function lies between $-\pi/2$ and $+\pi/2$).

A third issue is to remind students that *the inverse sine function is only defined for restricted values of x*. For example, at least in the context of first year calculus $\sin^{-1} 2$ is not meaningful. (You might want to think about which complex values of z solve the equation $\sin z = 2$, but this is not for the students!)

Finally, you should remind students that in mathematics courses that *radian measure is the default for angle measurement* even though degree measure is much easier to visualize. Among other reasons, radian measure make certain formulas in calculus easier to handle.

Working sample problems.   At this stage it is time to work some sample problems from the exercises in Section 5.5 of the text (the one treating the inverse sine function). One particular suggestion is a variant of Example 9 on page 452, say from exercises 93–99 on pages 453–454. The idea for these problems is to set the right hand side equal to y and then to solve for x in terms of y.