NAME: ______________________________

Mathematics 9C, Winter 2006, Examination 2

Point values are indicated in brackets. Describe clearly any formulas or results that you use.

IMPORTANT FORMULAS

$$e^x = \sum \frac{x^k}{k!} \qquad \log_e(1+x) = \sum_{k>0} \frac{(-1)^{k-1}x^k}{k}$$

$$\sin x = \sum \frac{(-1)^k x^{2k+1}}{(2k+1)!} \qquad \cos x = \sum \frac{(-1)^k x^{2k}}{(2k)!}$$

$$y' + p'(x)y = q(x) \implies e^{p(x)}y = \int e^{p(x)}\, q(x)\, dx$$

$$P' = r(M-P)P \implies P = \frac{M}{1 + C\,e^{-rMt}}$$

(Where C is a constant of integration)

Solutions to problems not taken verbatim from the book.

3. [*20 points*] Find the power series expansion for the following function:

$$\int_0^x \frac{\sin u}{u}\, du$$

SOLUTION.

We have

$$\sin u \quad = \quad \sum \frac{(-1)^k u^{2k+1}}{(2k+1)!}$$

and therefore we also have

$$\frac{\sin u}{u} \quad = \quad \sum \frac{(-1)^k u^{2k}}{(2k+1)!}$$

and term by term integration then yields the formula

$$\int_0^x \frac{\sin u}{u}\, du \quad = \quad \sum \int_0^x \frac{(-1)^k u^{2k}}{(2k+1)!}\, du$$

and we can rewrite the right hand side of this equation as

$$\sum \frac{(-1)^k x^{2k+1}}{(2k+1)! \cdot (2k+1)} \ .$$

6. [*20 points*] Suppose we are given a population growth satisfying the logistic equation $y' = \frac{1}{2}(400 - y)y$, so that the upper limit is 400. Find the exact solution of this differential equation with initial condition $y(0) = 2$ and find the time t at which the population reaches 399. You may leave your answer in a form involving logarithms.

SOLUTION.

The general solution to an equation of the given type is presented on the first page, and it is only necessary to use this solution in order to find all the required numbers. It follows immedately that $r = \frac{1}{2}$ and the limiting population $M = 400$. Therefore the solution to the differential equation in this example must have the form

$$y(t) \quad = \quad \frac{400}{1 \ + \ C\,e^{-200t}}$$

(since $rM = 200$). To find t we must use the initial condition $y(0) = 2$, obtaining the equation

$$2 \quad = \quad \frac{400}{1 + C}$$

and if one carries out the algebra to solve this one finds that $C = 199$. Therefore the exact solution is given by

$$y(t) \quad = \quad \frac{400}{1 \ + \ 199\,e^{-200t}} \ .$$

The second part of the problem asks for the value of t such that $y(t) = 399$. Thus we need to solve

$$399 \quad = \quad \frac{400}{1 \ + \ 199\,e^{-200t}}$$

for t. Clearing of fractions we obtain the equation

$$399 \ + \ (399 \cdot 199) \cdot e^{-200t} \quad = \quad 400$$

which simplifies to

$$(399 \cdot 199) \cdot e^{-200t} \quad = \quad 1$$

and if we multiply both sides by e^{200t} we obtain the equation

$$(399 \cdot 199) \quad = \quad e^{200t} \ .$$

Taking natural logarithms of both sides yields

$$\log(399 \cdot 199) \quad = \quad 200t$$

and if we divide both sides we obtain the equation

$$\frac{\log(399 \cdot 199)}{200} \quad = \quad t$$

which according to the instructions in the problem is an acceptable form for the answer (the numerical answer is 0.056411... to six decimal places).