AMBIGUITY PROBLEMS WITH POLAR COORDINATES

A given point in the plane can be represented by more than one pair of polar coordinates, and Section 9.5 of the text discusses to problems related to this fact.

(1) If we are given a curve C defined by an equation $F(r, \theta) = 0$ and a point P with polar coordinates (q, α), then P may lie on C even if $F(q, \alpha) \neq 0$.

(2) If we are given two curves A and B defined by equations $F(r, \theta) = 0$ and $G(r, \theta) = 0$, then there may be hidden points (q, α) lying on BOTH curves A and B such that (q, α) is NOT a simultaneous solution of the given two equations. In other words, at least one of $F(q, \alpha)$ or $G(q, \alpha)$ is nonzero.

Specific examples are given on pages 766 – 768 of the text, which addresses the second issue by suggesting that one graph the curves A and B to see if there are any common points that are not given by simultaneous solutions of the equations. This is usually effective, but it is not systematic or logically complete. We shall describe an analytic procedure for finding all such hidden points and use it to solve examples like those in the text.

Points with the same polar coordinates

Since the problems in (1) and (2) arise because a point is representable by more than one set of polar coordinates, it is best to begin by recalling when two pairs of polar coordinates (q, α) and (s, β) define the same point in the plane. There are two distinct criteria:

(i) Both q and s are nonzero, and there is an integer n such that $\beta = \alpha + n\,\pi$ and $s = (-1)^n q$.

(ii) Both q and s are zero, and α and β are arbitrary.

These criteria are exactly the conditions under which the‘ equations $q \cos \alpha = r \cos \alpha$ and $q \sin \alpha = r \sin \alpha$ are both satisfied.■

Application to ambiguity issues

Here is the general criterion for determining whether a point with polar coordinates (q, α) lies on the curve C defined by $F(r, \theta) = 0$.

General condition for (1): *If the curve C is defined by $F(r, \theta) = 0$, then the point P with polar coordinates (q, α) lies on C if and only if one of the following is true:* (a) *We have $q \neq 0$ and there is an n such that $F(\,(-1)^n q, \alpha + n\pi) = 0$.* (b) *We have $q = 0$ and there is some β such that $F(q, \beta) = 0$.*■

General condition for (2): *If the curves A and B are defined by $F(r, \theta) = 0$ and $G(r, \theta) = 0$ respectively, then the point P with polar coordinates (q, α) lies on both A and B if and only if one of the following is true:* (a) *We have $q \neq 0$ and there are integers m and n such that $F(\,(-1)^m q, \alpha + m\pi) = 0$ and $G(\,(-1)^n q, \alpha + n\pi) = 0$.* (b) *We have $q = 0$ and there are some β and γ such that $F(q, \beta) = 0$ and $F(q, \gamma) = 0$.*■

Textbook examples and exercises usually have a crucial property which leads to simplified criteria. In such examples the function $H = F$ and (if applicable) G satisfy the **periodicity property**

$$H(r, \theta) \quad = \quad H(r, \theta + 2\pi)$$

for all r and θ. In particular, this holds when H can be written as a sum of terms of the form $f(r)\sin^a\theta\cos^b\theta$ where a and b run through finite sets of integers (note that functions like $\sin k\theta$ and $\cos \ell\theta$ are all polynomials in $\sin\theta$ and $\cos\theta$). In such cases there are only two possibilities for the values of the expressions $H((-1)^n q, \alpha + n\pi)$ — namely, $H(q,\alpha)$ and $H(-q,\alpha+\pi)$ — and one can simplify criterion (a) in the general conditions as follows:

Specialized condition for (1): *If the curve C is defined by $F(r,\theta) = 0$ where F has the periodicity property, then the point P with polar coordinates (q,α) lies on C if and only if one of the following is true: (a) We have $q \neq 0$ and either $F(q,\alpha) = 0$ or $F(-q,\alpha+\pi) = 0$. (b) We have $q = 0$ and there is some β such that $F(q,\beta) = 0$.*■

Specialized condition for (2): *If the curves sA and B are defined by $F(r,\theta) = 0$ and $G(r,\theta) = 0$ respectively, then the point P with polar coordinates (q,α) lies on both A and B if and only if one of the following is true: (a) We have $q \neq 0$ and either $F(q,\alpha) = G(q,\alpha) = 0$ or $F(q,\alpha) = G(-q,\alpha+\pi) = 0$ or $F(-q,\alpha) = G(q,\alpha) = 0$. (b) We have $q = 0$ and there are some β and γ such that $F(q,\beta) = 0$ and $F(q,\gamma) = 0$.*■

The proofs of all these statements are fairly straightforward if one uses some basic concepts from a Discrete Mathematics course (for example, Mathematics 11), but the details will not be needed to work the examples we shall discuss.

Application to ambiguity issues

We shall now apply the preceding criteria to the examples worked out on pages 766 – 768 of the text and to solve exercises 67 – 68 on page 769 of the text.

EXAMPLE 8. We are given the curve $F(r,\theta) = r - 2\cos 2\theta = 0$ and we want to show that the point with polar coordinates $(2,\pi/2)$ lies on it. According to the first specialized condition, this will hold if either $F(2,\pi/2) = 0$ or $F(-2,3\pi/2) = 0$. The first of these does not hold, but the second does because $F(-2,3\pi/2)$ is equal to $F(-2,-\pi/2)$, and as noted in the text the latter turns out to be zero.■

EXAMPLE 9. We want to find the common points to the curves defined by $F(r,\theta) = r^2 - 4\cos\theta = 0$ and $G(r,\theta) = r - 1 + \cos\theta = 0$. According to the second specialized condition, there are four parts to this, the first three of which are solving the three pairs of simultaneous equations as in (a) and the last of which involves cases where $r = 0$.

One begins with the system of equations $F(r,\theta) = G(r,\theta) = 0$ as in the text, and this yields two common points. Next, one considers the system $F(r,\theta) = G(-r,\theta+\pi) = 0$, which reduces to $r^2 = 4\cos\theta$ and $-r = 1 + \cos\theta$. Following the same pattern as on page 767 of the text, we obtain the equation

$$-r \ = \ 1 + \cos\theta \ = \ 1 + \frac{r^2}{4}$$

which further reduces to $0 = r^2 + 4r + 1$, so that $r = \pm 2$. For these choices of r we have

$$\mp 2 \ = \ 1 + \cos\theta$$

so that $-1 \mp 2 = \cos\theta$. Since $-3 = \cos\theta$ is impossible, we are left with the case where $r = -2$ and $1 = \cos\theta$, so that the point with polar coordinates $(-2,0)$ also lies on the curve. Since the polar coordinates $(-2,0)$ and $(2,\pi)$ determine the same point, we conclude that $(2,\pi)$ is a hidden intersection point of A and B, as noted in the last paragraph of page 767.

The third part of the problem in our procedure is to examine the system $F(-r,\theta+\pi) = G(r,\theta) = 0$, and in this case the first equation is equivalent to $r^2 = -4\cos\theta$. If we square the second equation we obtain $r^2 = (1 - \cos\theta)^2$, and if we combine the preceding two equations we find that

$$-4\cos\theta \quad - \quad 1 \; - \; 2\cos\theta \; + \; \cos^2\theta$$

which is equivalent to $0 = (1 + \cos\theta)^2$. This implies $\cos\theta = 1$, so that $\theta = \pi$ and also $r = 0$. Strictly speaking, this means that we have no solutions corresponding to the third part of the problem because at this point we are only looking for solutions where the first polar coordinate is nonzero.

Finally, the fourth step is the one which decides whether or not the origin or pole lies on both curves. All we need to do is find α and β such that $F(0, \alpha) = 0$ and $G(0, \beta) = 0$. But we see directly that $F(0, \pi/2) = 0$ and $G(0, 0) = 0$, and therefore it follows that $(0, 0)$ is another hidden intersection point of A and B. Furthermore, this also shows that there are no other common points aside from the four that were found in the text.■

EXERCISE 67. Here we have the two cardioids $r = 1 \pm \cos\theta$. The curve with the negative sign is depicted on pages 764 and 767 of the text, and the curve with the positive sign is the mirror image of the first one with respect to the y-axis. Graphing these curves suggests that there are three common points, one at the origin and two others at points on the y-axis that are symmetric with respect to the origin. We need to show that our procedure yields all these points and no others.

The first step is to solve the simultaneous equations $0 = F(r, \theta) = r - 1 + \cos\theta$ and $0 = G(r, \theta) = r - 1 - \cos\theta$ when $r \neq 0$. These yield $\cos\theta = 0$ and hence that $\theta = \pi/2$ or $3\pi/2$ and consequently $r = 1$. Thus we have checked that the two points with polar coordinates $(1, \pm\pi/2)$ lie on the curve.

Next we consider the system $0 = F(r, \theta) = G(-r, \theta + \pi)$, The second equation is just $0 = -r - 1 + \cos\theta$, so we have the system of equations $r = 1 - \cos\theta = -1 + \cos\theta$. These yield the impossible equation $\cos\theta = 2$, so there are no hidden points given by simultaneous solutions of the second system.

Turing to the third system $0 = G(r, \theta) = F(-r, \theta + \pi)$, we have $F(-r, \theta + \pi) = -r - 1 - \cos\theta$, so this system becomes $r = -1 - \cos\theta = 1 + \cos\theta$. These yield the impossible equation $\cos\theta = -2$, so there are no hidden points given by simultaneous solutions of the third system.

Finally, we must check to see whether there exist α and β such that $F(0, \alpha) = G(0, \beta) = 0$. The answer is yes, and we can choose α and β to be 0 and π respectively.■

EXERCISE 68. Here we have the circle $r = 2\sin\theta$ whose radius is 1 and whose center has rectangular coordinates $(0, 1)$, and we also have the four leaf clover curve $r = 2\sin 2\theta$ whose center is the origin and whose four leaves each lie in a different quadrant of the coordinate plane (compare the curve at the bottom of page A – 54 of the text; the curve in the present problem is given by doubling all distances from the origin and rotating through a 45 degree angle).

Here the equations are $F(r, \theta) = r - 2\sin\theta$ and $G(r, \theta) = r - 2\sin 2\theta$. These yield the equation $\sin\theta = \sin 2\theta$, and if we solve these for θ we find that $\sin\theta = 0$ or $\cos\theta = \frac{1}{2}$. For the first option we have $\theta = 0$ or π, and for the second we have $\theta = \pm\pi/3$. The first option $\theta = 0$ or π implies $r = 0$, and although this is not strictly covered by the conditions in the first part of the procedure we can see that $F(0, 0) = G(0, 0) = 0$, so that $(0, 0)$ lies on both curves. This means we can skip the fourth part of the procedure. Turning to the second option, it implies that $r = \pm\sqrt{3}$, and therefore we also see that the points with polar coordinates $(\pm\sqrt{3}, \pm\pi/3)$ also lie on both curves.

The second part of the problem involves the equations $F(r, \theta) = G(-r, \theta + \pi) = 0$, which leads to the equation $\sin\theta = -\sin 2\theta$. The solutions to this equation are $\theta = 0$ or π and $\theta = \pm 2\pi/3$. We have already considered the first option, and the second option again leads to $r = \pm\sqrt{3}$, so that we obtain the points with polar coordinates $(\pm\sqrt{3}, \pm 2\pi/3)$ as common points of the curve.

However, these are not new because one can check that $(\pm\sqrt{3}, \pm 2\pi/3)$ represent the same points as $(\mp\sqrt{3}, \mp\pi/3)$ respectively.

Finally, the third part of the problem involves the equations $F(-r, \theta) = G(r, \theta) = 0$, which leads to the equation $-\sin\theta = \sin 2\theta$. This is equivalent to the equation considered in the second part of the problem, so it will yield nothing new. Therefore all the solutions are given by the (extended) first part of the problem as above.■