FOOTNOTE ON SEPARABLE DIFFERENTIAL EQUATIONS

As indicated in the text, the basic procedure for solving a separable (ordinary) differential equation is **(1)** rewrite it in the form

$$g(y)\,\frac{dy}{dx} \quad = \quad f(x)$$

(2) integrate f and g to obtain F and G such that $F' = f$ and $G' = g$, **(3)** solve the resulting equation $G(y) = F(x) + C$ (where C is a constant of integration) for y, **(4)** if we are also given the INITIAL VALUE CONDITION $y(a) = b$, finish by solving $G(b) = F(a) + C$ for C.

However, it is possible that one cannot complete the third step; the text does not cover this point. This normally does not cause problems, but for the sake of completeness we give a condition which ensures that everything works well and an example where the third step cannot be completed. In fact, there are many situations in which one can complete the third step even though the condition described below is not satisfied, and in standard textbook exercises this is almost always the case; the point here is similar to that of a warning label on a medication which states that the latter may not work properly if not used under the indicated conditions.

CONDITION FOR COMPLETING THE THIRD STEP. If we want to solve the given differential equation, then we need to solve the equation $z = G(y)$ for values of y that are close to the initial value b, and we need to do so uniquely. The standard way of ensuring this is **to require that $G'(b) = g(b)$ is not equal to zero.**

Under this condition we know that there is a well behaved inverse function h to G which is defined near $G(b)$, so we can then write the solution to the differential equation as $y = h(G(y)) = h(F(x) + C)$. — Note that we only claim that an inverse function h exists; it might not be possible to find a reasonable formula for h. However, this should be the case in all textbook problems.■

AN EXAMPLE. Suppose that we take $f(x) = 2x$ and

$$g(y) \quad = \quad \frac{d}{dy}\left(y^3 \sin \tfrac{1}{y}\right) \quad = \quad 3y^2 \sin \tfrac{1}{y} \; - \; y \cos \tfrac{1}{y} \; .$$

Then we have the separable differential equation $g(y)\,y' = f(x)$, and if we apply the second step of the solution procedure we obtain the following equation:

$$y^3 \sin \tfrac{1}{y} \quad = \quad x^2 \; + \; C$$

Suppose we want a solution of this such that $y(a) = 0$. In order to do this, it is necessary to know that the left hand side can be solved uniquely for y on some open interval of the form $-h < y < h$ where $h > 0$ is some sufficiently small number.

This is impossible because the expression on the left hand side is zero for infinitely many values of y close to 0; specifically, if $y_n = \frac{1}{n\pi}$ where n is any nonzero integer, then it follows that

$${y_n}^3 \sin(1/y_n) \quad = \quad 0$$

so there is no small interval of the form $-h < y < h$ on which one can solve $z = y^3 \sin(1/y)$ uniquely for y.■