WORKED OUT MIXTURE PROBLEM

This gives more details for the solution to Exercise 27 on page 507 of the text. The treatment is based upon the discussion on pages 502–503 of the text.

As noted on pages 502–503, when we have one fluid coming in and another going out, the total rate of change of the amount of chemical in the tank is the rate $\mathrm{IN}(t)$ at which the chemical is being pumped in one end minus the rate $\mathrm{OUT}(t)$ at which the chemical is being drained out the other. Let $y(t)$ be the amount at time t and let $V(t)$ be the total volume in the tank at time t. Then we clearly have

$$y'(t) \quad = \quad \mathrm{IN}(t) \ - \ \mathrm{OUT}(t) \ .$$

It is easy to compute $\mathrm{IN}(t)$ because the fluid is entering the tank at a rate of 1 gallon per minute and there is one pound of chemical per gallon in the incoming mixture. Therefore $\mathrm{IN}(t) = 1$.

Computing $\mathrm{OUT}(t)$ requires more work. It is given by the concentration of the chemical at time t multiplied by the rate at which fluid is being drained out the second end. The concentration is just the among times the volume, so it is just $y(t)/V(t)$, and the outgoing rate is 3 gallons per minute, so $\mathrm{OUT}(t)$ is equal to $3y(t)/V(t)$.

In order to finish setting up the differential equation we need to have an explicit formula for $V(t)$. Now fluid is entering one end at a rate of 1 gallon per minute and draining out the other end at a rate of 3 gallons per minute, so the net effect is that the volume is decreasing at a rate of 2 gallons per minute. Since the tank initially holds 100 gallons, this means that $V(t) = 100 - 2t$.

The preceding discussion then yields the differential equation

$$y' \quad = \quad 1 \ - \ \frac{3y}{100 - 2t}$$

which we rewrite in the standard linear form as

$$y' \ + \ \frac{3y}{100 - 2t} \quad = \quad 1 \ .$$

This has the form $y' - p'(x)y = 1$ where $p' = 3/(100 - 2t)$. The next step is to evaluate $p(x) = \int p'(x)\,dx$, and if we do so we see that $p(x)$ is equal to $\frac{-3}{2}\log(100 - 2t)$ plus a constant that we can ignore at this stage. We then have the general formula

$$(e^{p(x)}y)' \quad = \quad e^{p(x)}\,y' \ + \ p'(x)e^{p(x)}\,y \quad = \quad e^{p(x)}$$

and if we substitute for p and p' as above we see that $e^{p(x)} = (100 - 2t)^{-3/2}$ and therefore

$$\left[(100 - 2t)^{-3/2}y\right]' \quad = \quad (100 - 2t)^{-3/2}$$

and if we integrate both sides of this equation we obtain the following equation:

$$(100 - 2t)^{-3/2}y \quad = \quad (100 - 2t)^{-1/2} \ + \ C$$

Finally we divide by the coefficient of y on the left hand side to obtain the general solution to the original differential equation:

$$y \quad = \quad (100 - 2t) \ + \ C\,(100 - 2t)^{-3/2}$$

We know that the tank initially contains none of the chemical, and this translates into the initial value condition $y(0) = 0$. Therefore we may solve for the constant of integration C:

$$0 \quad = \quad 100 \; + \; C\,(100)^{-3/2}$$

The solution to this equation is $C = -1/\sqrt{10}$, so now we can write down the function $y(t)$ explicitly:

$$y \quad = \quad (100 - 2t) \quad - \quad \frac{1}{\sqrt{10}} \cdot (100 - 2t)^{-3/2}$$

In the lectures we did not look at the final part of the problem (when is the amount of chemical in the tank maximized). This can be found by solving the equation $y'(t) = 0$ for t. As indicated in the solutions at the back of the text, this equation yields a value of approximately 27.8 minutes.