AN EXAMPLE INVOLVING POWER SERIES

One can use either the Ratio Test or the Root Test to show that the series

$$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

converges. In fact, the convergence of this series and its value were known in the 14th century. We shall describe how one can compute the value of this sum using power series. The original calculations of the sum used other methods.

We start with the geometric series

$$\sum_{n=0}^{\infty} x^n \quad = \quad \frac{1}{1-x}$$

which we know is valid for $|x| < 1$. We also know that we can differentiate power series term by term to obtain the derivative of the original infinite series. For the geometric series this yields the following formula:

$$\sum_{n=1}^{\infty} n\, x^{n-1} \quad = \quad \frac{1}{(1-x)^2}$$

If we multiply both sides by x we obtain the following equation, which by the termwise differentiation rule is valid for $|x| < 1$:

$$\sum_{n=1}^{\infty} n\, x^{n} \quad = \quad \frac{x}{(1-x)^2}$$

The series we want to compute is given by substituting $x = 1/2$ into the left hand side, so its value is given by substituting the same value into the right hand side. However evaluation of the right hand side at $x = 1/2$ yields a value of 2, and therefore we have computed the sum in question:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \quad = \quad 2$$