# SOULS AND MODULI SPACES OF NONNEGATIVELY CURVED MANIFOLDS

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**<u>PROBLEM.</u>** **Which smooth manifolds support complete riemannian metrics with everywhere nonnegative sectional curvature? If so, then how many "substantially different" such metrics are there?**

**(Emphasis is on the $2^{nd}$ part here)**

**<u>CHEEGER – GROMOLL SOUL THEOREM.</u>** ***If a noncompact, connected manifold admits such a metric, then it is diffeomorphic to the total space of a vector bundle over a compact manifold which admits such a metric.***

**The compact manifold is totally geodesic and is called a <u>SOUL</u> of the original riemannian manifold. The fiber dimension is called the soul's <u>codimension</u>.**

**HOW UNIQUE IS THE SOUL?**

**A given metric can have many souls. Consider $S^n \times \mathbb{R}^k$ with the product metric (constant positive curvature on the first factor, flat on the second). Then every first factor slice of the form $S^n \times \{point\}$ is a soul.**

**However, a result of V. Sharafutdinov implies uniqueness up to ambient isotopy.**

**But *different* metrics can have non – diffeomorphic souls. For example, there are many pairs of non – diffeomorphic lens spaces $L_1$ and $L_2$ such that $L_1 \times \mathbb{R}^3$ and $L_2 \times \mathbb{R}^3$ are diffeomorphic (*cf.* Milnor's first counterexamples to the *Hauptvermutung*). — Incidentally, a result of S.K. + R.S. implies that one cannot replace $\mathbb{R}^3$ by $\mathbb{R}^2$.**

**DEFAULT ASSUMPTION. To simplify the discussion, restrict to the simply connected case.**

**[There are also some results, particularly for small fundamental groups.]**

**In fact, there are also simply connected examples with non – diffeomorphic souls. Several exotic 7 – spheres $\Sigma^7$ admit metrics with nonnegative sectional curvature, and it is well – known that $\Sigma^7 \times \mathbb{R}^3$ and $S^7 \times \mathbb{R}^3$ are diffeomorphic for all exotic 7 – spheres $\Sigma^7$. As before, one cannot replace $\mathbb{R}^3$ by $\mathbb{R}^2$ (this actually goes back to the nineteen sixties).**

**<u>ENUMERATION PROBLEM.</u> How many distinct diffeomorphism types of souls can be realized by different (nonnegatively curved) metrics on the same manifold?**

**<u>Easy case.</u> For simply connected souls of codimension 1 and dimension at least 5, the $h$ – cobordism Theorem implies that only one type can be realized.**

**However, in some cases with larger codimensions, infinitely many types can be realized (I.B.).**

**Subsequently, Kapovitch – Petrunin – Tuschmann gave other such examples with better geometric properties and also showed that sufficiently close (nonnegatively curved) metrics have diffeomorphic souls.**

**<u>THEOREM 1.</u>** ***In fact, sufficiently close metrics have (smoothly) ambiently isotopic souls.***

**<u>REFINED QUESTION.</u>** **How does the answer to the enumeration problem vary with the dimension and codimension of the soul?**

**Restrict attention further to souls of dimension at least 5.**

**<u>THEOREM 2.</u>** ***Let $E$ be a riemannian manifold with a complete metric of nonnegative sectional curvature, and suppose that the codimension of the soul is $\leq 3$. Then there are only finitely many diffeomorphism classes of smooth manifolds which can be souls of complete, nonnegatively curved metrics on $E$.***

**The codimension hypothesis in this result is best possible.**

**<u>THEOREM 3.</u>** ***For every n > 6, there are compact nonnegatively curved n – manifolds $M^n$ such that $M^n \times \mathbb{R}^4$ supports infinitely many complete nonnegatively curved metrics with pairwise nondiffeomorphic souls.***

**By crossing with $\mathbb{R}$ (equipped with the usual flat metric), *<u>we can also obtain examples in all higher codimensions</u>*. The examples of both Belegradek and Kapovitch – Petrunin – Tuschmann are in fact products of the examples in the theorem with sufficiently many copies of $\mathbb{R}$.**

**<u>IDEAS OF PROOFS.</u>** ***Low codimensions*: Codimension 1 was already mentioned.**

**In codimension 2, the bundle is determined by its Euler class, which is a fiber homotopy invariant, and this implies that all possible souls are tangentially homotopy equivalent (the stable tangent bundle pulls back under the homotopy equivalence). By surgery theory, there are only finitely many diffeomorphism classes of closed simply connected manifolds in a given tangential homotopy type.**

**Codimension 3 can be handled similarly using the fact that *the first rational Pontryagin class is a fiber homotopy invariant for* 3 – *plane bundles* (but NOT for higher fiber dimensions!!). Since there are only finitely many 3 – plane bundles with a given first rational Pontryagin class, one can proceed much as in the codimension 2 case.**

***Codimension* 4: The earlier infinite families of examples were constructed by showing that the candidates for souls could be smoothly embedded in certain products of spheres with $\mathbb{R}^k$ for suitable values of $k$. We use an embedding theorem of Browder – Casson – Haefliger – Sullivan – Wall to show that in the earlier examples one can always take $k$ equal to 4.**

**Here is another type of exotic example in codimension 4:**

**THEOREM 4.** ***There is a complete, nonnegatively curved metric on $S^7 \times \mathbb{R}^4$ such that the soul is $S^7$ but the normal bundle of the soul is nontrivial.***

**By Theorem 1, this metric and the usual product metric must belong to separate components in the moduli space of complete nonnegatively curved metrics on $S^7 \times \mathbb{R}^4$.**

**<u>A REMAINING QUESTION.</u> In the low codimension cases, is it possible to find examples of metric pairs for which the souls are not diffeomorphic? By previous remarks, the only possibilities are codimensions 2 and 3 (and as usual a positive answer in the first case implies the a positive answer in the second).**

**<u>Candidates for souls.</u> Consider the manifolds $\Sigma^7 \times \mathbb{CP}^{2k}$ where the first factor is either the standard 7 – sphere or an exotic 7 – sphere. Results of K. Grove and W. Ziller imply that many such product manifolds admit nonnegatively curved metrics, and in particular this holds if $\Sigma^7$ generates the Kervaire – Milnor group $\Theta_7$ of homotopy 7 – spheres, which is cyclic of order 28.**

**Classifying such products up to diffeomorphism is "an interesting problem" with a long history.**

**NOTE.** We restrict to even – dimensional complex projective spaces because $\Sigma^7 \times \mathbb{CP}^{2k+1}$ and $S^7 \times \mathbb{CP}^{2k+1}$ are always diffeomorphic.

**OBSERVATION.** If $\xi$ is a $2$ – plane bundle over $\mathbb{CP}^q$ then $E(\xi)$ has a complete positively curved metric whose soul is the zero section. — The same will be true for many (and, for most values of $q$, all) products $\Sigma^7 \times E(\xi)$.

Examples of $2$ – plane bundles with nondiffeomorphic souls follow from the next two results.

**THEOREM 5.** *If the $2$ – plane bundle $\xi$ is nontrivial, then $\Sigma^7 \times E(\xi)$ and $S^7 \times E(\xi)$ are always diffeomorphic.*

Note that the conclusion is FALSE if $\xi$ is trivial!

**THEOREM 6.** *If $\Sigma^7$ generates the Kervaire – Milnor group of homotopy $7$ – spheres, then for each $k > 0$ the manifolds $\Sigma^7 \times \mathbb{CP}^{2k}$ and $S^7 \times \mathbb{CP}^{2k}$ are not diffeomorphic.*

**PROOFS.** **Both involve surgery theory. The first is a fairly straightforward application of the Sullivan – Wall exact surgery sequence. The second requires a much deeper study of exact surgery sequences, including detailed results on which homotopy self – equivalences of $S^7 \times \mathbb{CP}^{2k}$ are homotopic to diffeomorphisms and a variant of a key result originally due to L. Taylor (some key parts of this go back to a nineteen eighties paper by R.S.).**

**THE SPECIAL CASE WHERE $k = 1$.** **The product manifolds $\Sigma^7 \times \mathbb{CP}^2$ yield exactly three diffeomorphism classes, and in fact every smooth structure on $S^7 \times \mathbb{CP}^2$ is diffeomorphic to one of these three products (so every smooth structure on the latter supports a nonnegatively curved metric).**