THE L-THEORY OF GROUPS WITH PERIODIC COHOMOLOGY I.

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Introduction.

In this paper and in a second part II we calculate the intermediate L-groups $L_n'(\pi)$ of finite groups π with periodic cohomology groups. The results imply via the surgery exact sequence a classification of spherical space forms up to topological weak s-cobordism, and in many cases up to homeomorphisms.

The paper is an application of the techniques developed by C.T.C. Wall in a series of papers for calculating $L_n'(\pi)$. Recall that these groups are determined from the groups $L_n^X(\mathbb{Z}\pi)$ which fit into the exact sequence

0.1 $$\ldots \xrightarrow{\psi_{n+1}} L_{n+1}^X(\hat{Q}\pi) \xrightarrow{\partial} L_n^X(\mathbb{Z}\pi) \xrightarrow{\rho_n} \Pi L_n^X(\mathbb{Z}_p\pi) \times L_n^X(Q\pi) \xrightarrow{\psi_n} \ldots$$

Here $\mathbb{Z}_p$ denotes the p-adic integers and $\hat{Q}$ is the ring of finite adèles, $\hat{Q} = Q \otimes \Pi\mathbb{Z}_p$. The groups $L_n^X(Q\pi)$, $L_n^X(\hat{Q}\pi)$ and $L_n^X(\mathbb{Z}_p\pi)$, $p \neq 2$ are easy to describe in terms of the (rational) representation theory of π, and the same it the case for the maps in 0.1. The difficult (and 'irregular') part is the description of

0.2 $$\bar{\psi}_n: L_n^X(\mathbb{Z}_2\pi) \to L_n^X(Q_2\pi).$$

There is the Rothenberg exact sequence

$$\ldots \to H^{n+1}(\mathbb{Z}/2; K_1'(\mathbb{Z}_2\pi)) \to L_n^X(\mathbb{Z}_2\pi) \to L_n^K(\mathbb{Z}_2\pi) \to \ldots$$

and since $L_n^K(\mathbb{Z}_2\pi) \cong L_n^K(\mathbb{Z}_2\pi/\mathrm{rad})$ is easy, the calculation of $L_n^X(\mathbb{Z}_2\pi)$ is essentially reduced to the calculation of $H^*(\mathbb{Z}/2; K_1'(\mathbb{Z}_2\pi))$. In general, $K_1'(\mathbb{Z}_2\pi)$ and its Tate cohomology groups can be determined quite easily from [O3], but in this description the map $\bar{\Psi}_n$ in 0.2 cannot be calculated: one needs to know explicitly the image of

$$\mathrm{Nrd}: K_1'(\mathbb{Z}_2\pi) \to C(Q_2\pi)^*$$

of the reduced norm in the center of $Q_2\pi$.

Without loss of generality one may assume π is 2-hyper-elementary, $\pi = \mathbb{Z}/n \tilde{\times} \sigma$. Then $\mathbb{Z}_2\pi = \prod_{d|n} \mathbb{Z}_2[\zeta_d]^t\sigma$. In this paper we consider only those groups with periodic cohomology whose 2-hyperelementary subgroups decompose into twisted group rings $\mathbb{Z}_2[\zeta_d]^t\sigma$ for which $t: \sigma \to \mathbb{Z}/d^*$ is either trivial or has abelian kernel. If t is trivial,

$$\bar{\Psi}_n: L_n^X(\mathbb{Z}_2[\zeta_d]^t\sigma) \to L_n^X(Q_2[\zeta_d]^t\sigma)$$

can be reduced to the case where $d = 1$ (already treated in [W6]); if t is non-trivial then $\hat{\mathbb{Z}}_2[\zeta_d]^t$ is Morita equivalent to its center, and we are reduced to an abelian (but not trivial) situation.

The groups with periodic cohomology which have the above 2-hyperelementary subgroup structure include all groups of types I, III and V in the usual classification, and part of the groups of type II, IV and VI. For example $S\ell_2(\mathbb{F}_p)$ and its degree 2 extension $T\ell_2(\mathbb{F}_p)$ are included. The remaining periodic groups are treated in part II.

The paper is divided into six sections and an appendix:

Detailed comments about the content of each section is not suitable but we point out that §3 contains a general discussion of $L_n^X(\mathbb{Z}_2\pi)$ which might be of interest in other connections. We prove for example in Theorem 3.6 that when $(\rho:1)$ is odd then

$$L_i^X(\mathbb{Z}[\pi \times \rho]) \cong L_i^X(\mathbb{Z}\pi) \oplus \Sigma$$

where Σ is a torsion free signature group, generalizing a result from [W6]. In §4 we give the main technical calculation of the paper, correcting an error in [LM1].

If a periodic group π contains subgroups of the form $\mathbb{Z}/d \times Q8$ then $SK_1(\mathbb{Z}\pi) \neq 0$ and otherwise $SK_1(\mathbb{Z}\pi) = 0$ by results from [O3]. In a planned part III we study the exact sequence

$$\ldots \to H^{n+1}(\mathbb{Z}/2, SK_1(\mathbb{Z}\pi)) \to L_n^S(\pi) \to L_n'(\pi) \to H^n(\mathbb{Z}/2; SK_1(\mathbb{Z}\pi)) \to \ldots$$

§1. Rational group rings of periodic groups.

It is well known that a periodic 2-group is either cyclic or quaternion. Thus if we denote by $Q2^k$ the quaternion group of order 2^k,

$$Q2^k = \langle X,Y \mid X^{2^{k-2}} = Y^2,\ YXY^{-1} = X^{-1}\rangle ,$$

there are two types of 2-hyperelementary periodic groups,

1.1
$$\text{I:}\quad 1 \to \mathbb{Z}/n \to \tau \to \mathbb{Z}/2^k \to 1$$
$$\text{II:}\quad 1 \to \mathbb{Z}/n \to \tau \to Q2^k \to 1$$

where $\mathbb{Z}/n$ is an odd order cyclic normal subgroup of τ. The groups in case II are divided up further into cases according to the kernel of the action $t\colon Q2^k \to (\mathbb{Z}/n)^\times$ specifying the group.

1.2
IIA $\mathrm{Ker}(t\colon Q2^k \to (\mathbb{Z}/d)^\times)$ is abelian for all $d|n$, $d>1$.
IIB $\mathrm{Ker}(t\colon Q2^k \to (\mathbb{Z}/d)^\times)$ is non-abelian for some $d>1$, $d|n$.

Note that these classes are closed under taking subgroups Every group ring $\Lambda\pi$ has a canonical anti-involution which we always denote by α, $\alpha(\Sigma n_g g) = \Sigma n_g g^{-1}$.

In this paragraph we classify the antistructures $(Q\tau,\alpha,1)$ for the groups in 1.1. Later in the paper we need to consider antistructures over certain adic group rings. First we recall standard facts and terminology.

Let R be an Azumaya algebra over A (i.e. separable and central, cf. [DI]). We consider antistructures (R,α,u) with $u \in A^\times$ $(u\alpha(u)=1)$. Two such (R,α,u) and (R,β,v) are called scaling equivalent, $(R,\alpha,u) \simeq (R,\beta,v)$ if there exists $r \in R^\times$ with $r\alpha(r)^{-1} \in A$ such that

1.3
$$\beta(x) = r\alpha(x)r^{-1},\quad v = r\alpha(r)^{-1}u$$

(Multiplying a hermitian form over (R,α,u) by $r \in R^\times$ gives a hermitian form over (R,β,v) and this correspondance is bijective). The restrictions of α to the center A induces a $\mathbb{Z}/2$-action on $A^\times$. We let $H^i(A^\times)$ denote the corresponding Tate cohomology groups. The following lemma covers all cases of interest to us.

Lemma 1.4. Suppose the Picard group $\mathrm{Pic}(A) = 0$.

(i) Any two anti-involutions on R which agree on A are conjugate.

(ii) For fixed anti-involution α, the equivalence classes of antistructures (R,α,u) are enumerated by $\mathrm{cls}(u) \in H^1(A^\times)$.

Proof. If α and β are anti-involutions on R with the same restriction to A then $\alpha\beta$ is a central automorphism of R. This is inner when $\mathrm{Pic}(A) = 0$ ([DI], Theorem 6.2), say $\alpha\beta(x) = rxr^{-1}$. Then $\beta(x) = \alpha(r)^{-1}\alpha(x)\alpha(r)$, and since $\beta^2(x) = x$ it follows that $\alpha(r)^{-1}\cdot r \in A^\times$. This proves (i).

We have $(R,\alpha,u) \simeq (R,\alpha,v)$ if and only of there exists $a \in R^\times$ with $\alpha(x)a = a\alpha(x)$ for all x and $v = a\alpha(a)^{-1}u$. Thus $a \in A^\times$ and $\mathrm{cls}(v) = \mathrm{cls}(u)$ in $H^1(A^\times)$. //

As a special case, consider $R = M_n(D)$ where D is a division algebra over the field K. Let $c = \alpha|K$. Then $H^1(K^\times) = 0$ if $c|K \neq 1_K$ by Hilbert's Theorem 90, and $H^1(K^\times) = \langle\pm 1\rangle$ if $c|K = 1_K$. The three possible equivalence classes of antistructures in this case are called types. More precisely, the antistructure $(R,\alpha,1)$, $R = M_m(D)$ has type U, if $\alpha|K \neq 1_K$

1.5 type O, if $\alpha|K = 1_K$ and $\dim_K(S^\alpha) = \frac{n^2+n}{2}$

type Sp, if $\alpha|K = 1_K$ and $\dim_K(S^\alpha) = \frac{n^2-n}{2}$

where $n^2 = \dim(R:K)$. For $(R,\alpha,-1)$ the roles of type O and type Sp are interchanged.

Finally, for any ring S we have an indecomposable antistructure $(S \oplus S,\alpha,u)$ where α interchanges the two copies of R. We say this antistructure has type GL,

1.5a type GL, if $R \cong S \oplus S$ and $\alpha(s_1,s_2)=(s_2,s_1)$

Given a finite group τ, $Q\tau$ decomposes into a product $Q\tau = \Pi M_{n(i)}(D_i)$. The standard anti-involution α on $Q\tau$ preserves each component. Indeed, if this was not the case then there would exist an idempotent e_i (associated to the above decomposition) with $e_i\alpha(e_i) = 0$. But this contradicts that for $x = \Sigma n_i g_i$,

1.6 $$Tr_{Q\tau/Q}(x\alpha(x)) = |\tau|\ \Sigma n_i^2 > 0.$$

Thus, in the decomposition of $(Q\tau,\alpha,u)$, type GL does not occur. However, for adic group rings type GL often is present in the decomposition

Returning to the setting of 1.4 we can associate to each (R,α,u) a matrix antistructure $(M_n(R),\alpha_n,u_n)$ where $\alpha_n((r_{ij})) = (\alpha(r_{ji}))$ and $u_n = uI_n$.

Definition 1.7. In the setting of 1.4, two antistructures (R,α,u) and (R',α',u') are called Morita equivalent if one is

scaling equivalent (in the sense of 1.3) to a matrix antistructure on the other; (in sign: $(R,\alpha,u) \sim (R',\alpha',u')$).

The simple components of group algebras $Q\tau$ can be conveniently described as crossed product algebras, cf. [R, section 29]. Let L/E be a field extension with finite Galois group G. Suppose $f\colon G \times G \to L^\times$ is a 2-cocycle. The associated algebra $S = (L/E,f)$ is the vector space $S = \sum_{\sigma\in G} Lu_\sigma$ with multiplication

1.8 $$u_\sigma x u_\sigma^{-1} = \sigma(x),\quad u_\sigma u_\tau = f_{\sigma,\tau}u_{\sigma\tau};$$

where $x \in L$ and $\sigma,\tau \in G$. As a special case we have the cyclic algebras when G is cyclic of order n.

1.9 $$(L/E,X,a) = L^t[X|X^n = a],\quad X\ell X^{-1} = X(\ell).$$

We have $(L/E,X,a) \cong M_n(E)$ if and only if $a \in E^\times$ is a norm from $L^\times$. Another special case is the twisted group rings $L^t[G]$ where $f_{\sigma,\tau} = 1$ for all $\sigma,\tau \in G$.

For each 2-hyperelementary group $\tau = \mathbb{Z}/n \rtimes \sigma$, $t\colon \sigma \to (\mathbb{Z}/n)^\times$, the decomposition of $Q[\mathbb{Z}/n]$ into a product of fields induces a decomposition

$$(Q\tau,\alpha,1) = \prod_{d|n} (S(d),\alpha,1)$$

$$S(d) = Q(\zeta_d)^t[\sigma].$$

The twisted group ring is determined by

$$t\colon \sigma \to (\mathbb{Z}/n)^\times \to (\mathbb{Z}/d)^\times = Gal(Q(\zeta_d)/Q).$$

We now give the decomposition of $(S(d),\alpha,1)$ into its simple components for each group in 1.1.

<u>Proposition 1.11</u>. For τ of type I,

$$S(d) = \prod_{i=0}^{k-\ell} (Q(\zeta_{2^i d})/E_i, X, \zeta_{2^i})$$

where $E_i = Q(\zeta_{2^i d})^H$, $H = \mathrm{Image}(\mathbb{Z}/2^k \xrightarrow{t} (\mathbb{Z}/d)^\times) = \mathbb{Z}/2^\ell$. The types of the induced antistructures are

(i) $d=1$: type U, for $i \geq 2$; type O for $i = 0,1$

(ii) $d > 1$, $-1 \notin H$: all summands have type U

(iii) $d > 1$, $-1 \in H$: type U for $i \geq 2$; type Sp for $i = 1$; type O for $i = 0$.

<u>Proof</u>. Let X generate $\mathbb{Z}/2^k$ (and its image H). Then

$$S(d) = Q(\zeta_d)^t[X|X^{2^k}=1] \cong \prod_{i=0}^{k-\ell} Q(\zeta_{2^i d})^t[X|X^{2^\ell} = \zeta_{2^i}].$$

The group H embeds into $(\mathbb{Z}/d)^\times \subset (\mathbb{Z}/2^i d)^\times$ and each factor is a cyclic algebra

$$Q(\zeta_{2^i d})^t[X|X^{2^\ell} = \zeta_{2^i}] = (Q(\zeta_{2^i d})/E_i, X, \zeta_{2^i}).$$

The types are U if $i \geq 2$ or $-1 \notin H$. Indeed $E_i = Q(\zeta_{2^i})(\theta)$ where $\theta = \Sigma \zeta_d^{a^i}$ if $X\zeta_d X^{-1} = \zeta_d^a$, and

$$\alpha(\zeta_{2^i}) = \zeta_{2^i}^{-1} \neq \zeta_{2^i} \quad \text{if} \quad i \geq 2$$

$$\alpha(\theta) = \Sigma(\zeta_d^{a^i})^{-1} \neq \theta \ \text{if} -1 \notin H = \langle a \rangle. \ //$$

For type II groups we have Milnor's description:
Write $\mathbb{Z}/n = \mathbb{Z}/a_1 \times \mathbb{Z}/a_2 \times \mathbb{Z}/a_3 \times \mathbb{Z}/a_4$ (coprime factors) such that $t\colon Q2^k \to (\mathbb{Z}/n)^\times$ is given by

$$t(X) = (1,1,-1,-1), \ t(Y) = (1,-1,1,-1).$$

The resulting group is denoted

1.12 $$\mathbb{Z}/n \tilde{\times} Q2^k = \mathbb{Z}/a_1 \times Q(2^k a_2; a_3, a_4)$$

We can permute the roles of a_3 and a_4 by interchanging Y with XY in the presentation of $Q2^k$, and may thus assume $a_3 \geq a_4$. (If $k=3$ we can even assume $a_2 \geq a_3 \geq a_4$). The groups of type IIA in 1.2 corresponds to $a_3 = a_4 = 1$ or to $k = 3$; the others have type IIB. We show in §3 that L-groups of $\mathbb{Z}/a \times \pi$ (a odd) are easily expressed in terms of those of π. Thus we can restrict attention to the groups in 1.12 with $a_1 = 1$. We begin with the metacyclic group $Q(2^k a_2) = Q(2^k a_2; 1,1)$

<u>Proposition 1.13</u>. For $\tau = Q(2^k a_2)$

$$S(d) = \mathbb{H}(E_{k-1}) \times \prod_{i=0}^{k-2} M_2(E_i) \quad \text{for} \quad d > 1$$

$$S(1) = \mathbb{H}(E_{k-1}) \times \prod_{i=2}^{k-2} M_2(E_i) \times Q^4 \quad \text{for} \quad d = 1,$$

and $E_i = Q(\zeta_{2^i d} + \zeta_{2^i d}^{-1})$. The types are Sp for the quaternion algebras and O for the others.

<u>Proof</u>. We have $S(d) = Q(\zeta_d)^t[Q2^k]$ and use inductively the idempotents $\frac{1}{2}(1 \pm X^{2^i})$, $i \leq k-2$, to get

$$S(d) = Q(\zeta_d)[\mathbb{Z}/2^{k-1}]^t[Y] = \prod_{i=0}^{k-1} Q(\zeta_{2^i d})^t[Y]$$

where $Y^2 = X^{2^{k-2}} = (\zeta_{2^i})^{2^{k-2}}$. Hence $Y^2 = 1$ for $i < k-1$ but $Y^2 = -1$ for $i = k-1$. With the notation of 1.9,

$$Q(\zeta_{2^i d})^t[Y] = (Q(\zeta_{2^i d})/E_i, Y, \pm 1), \quad i = 0, \ldots, k-1.$$

If $d=1$, the components with $i = 0,1$ split further, giving the 4 copies of $\mathbb{Q}$. Note that the component with $i = k-1$, $Y^2 = -1$ is isomorphic to $\mathbb{H}(E_{k-1})$. The other components are matrix rings over their centers. //

Let $\tau = \mathbb{Q}(2^k a_2; a_3, a_4)$; we have $\mathbb{Q}\tau = \Pi S(abc)$ where $a|a_2$, $b|a_3$ and $c|a_4$. Consider first the summands where at least two of the numbers a, b and c are larger than 1 (and $b \geq c$). Then

$$S(abc) = \mathbb{Q}(\zeta_{abc})^t[\mathbb{Q}2^k]$$

$$= \mathbb{Q}(\zeta_{2^{k-2}abc})^t[X,Y|X^2 = \zeta_{2^{k-2}}, Y^2 = -1, YXY^{-1} = X^{-1}] \times \prod_{i=0}^{k-3} \mathbb{Q}(\zeta_{2^i abc})^t[X,Y|X^2 = \zeta_{2^i}, Y^2 = 1, YXY^{-1} = X^{-1}]$$

Each component is a crossed product algebra with group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and center $E_i = \mathbb{Q}(\zeta_{2^i abc})^{\langle X,Y\rangle}$. If all numbers a,b and c are larger than 1, then E_i is always totally complex:

1.14a $\qquad E_i = \mathbb{Q}(\zeta_{2^i a} + \zeta_{2^i a}^{-1}, \zeta_b + \zeta_b^{-1}, \zeta_c + \zeta_c^{-1}, \theta)$

where $\theta = (\zeta_{2^i a} - \zeta_{2^i a}^{-1})(\zeta_b - \zeta_b^{-1})(\zeta_c - \zeta_c^{-1})$ is purely imaginary.

However,

1.14b $\qquad E_i = \mathbb{Q}(\zeta_{2^i a} + \zeta_{2^i a}^{-1}, \zeta_b + \zeta_b^{-1})$, $\quad c = 1$

1.14c $\qquad E_i = \mathbb{Q}(\zeta_b + \zeta_b^{-1}, \zeta_c + \zeta_c^{-1})$, $\quad a = 1$ and $i = 0,1$.

are real fields.

The types of the induced antistructures are easily determined by dimensions counts and we have

Proposition 1.15. Suppose that at least 2 of the numbers a, b and c are larger than one, and $b \geq c$. Then

$$S(abc) = \prod_{i=0}^{k-2} S_i$$

where S_i is a simple algebra with center 1.14.

(i) $a,b,c > 1$: type U for all i

(ii) $c = 1$: type Sp for $i = k-2$; type O for $i < k-2$

(iii) $a = 1$: type U for $i \geq 2$; type Sp for $i = 1$; type O for $i = 0$.

Remark 1.16. A cyclic algebra $(L/K,X,a)$ is split (alias equal to a matrix algebra over its center) if and only if a is a norm from L. For crossed-product algebras in general one must check if the cocycle defining it is a coboundary. However, in the cases above (with group $\mathbb{Z}/2 \oplus \mathbb{Z}/2$) it is often simpler to re-write the algebra as a tensor product of cyclic algebras. Here are two examples:

(i) $\mathbb{Q}(\zeta_{2^i ab})^t[X,Y|X^2 = \zeta_{2^i}, Y^2 = 1, YXY^{-1} = X^{-1}] \cong$

$$E_i(\zeta_b)^t[\bar{X}|\bar{X}^2 = \pi] \otimes_{E_i} E_i(\zeta_{2^i a})^t[Y|Y^2 = \pm 1], \quad 0 \leq i \leq k-2, \ i \neq 1$$

where $\bar{X} = (1+\zeta_{2^i}^{-1})X$, $\pi = 2 + \zeta_{2^i} + \zeta_{2^i}^{-1}$ and $E_i = \mathbb{Q}(\zeta_{2^i} + \zeta_{2^i}^{-1})$.

(ii) $\mathbb{Q}(\zeta_{ab})^t[X,Y|X^2 = -1, Y^2 = \pm 1, YXY^{-1} = -X] \cong$

$$E_1(\zeta_b)^t[\bar{X}|\bar{X}^2 = -(\zeta_a - \zeta_a^{-1})^2] \otimes_{E_1} E_1(\zeta_a)^t[Y|Y^2 = \pm 1],$$

where $\bar{X} = (\zeta_a - \zeta_a^{-1})X$. (Such rewritings make it easy to calculate Hasse invariants).

Proposition 1.17. For the groups $\tau = Q(2^k; a_3, 1)$ each factor $S(d)$, $d > 1$, is given by

$$S(d) = \prod_{i=2}^{k-2} A_i \times M_2(Q(\zeta_d)) \times M_2(Q(\zeta_d+\zeta_d^{-1})) \times M_2(Q(\zeta_d+\zeta_d^{-1})).$$

The simple algebras $S_i = Q(\zeta_{2^i d})^t[X,Y|X^2=\zeta_{2^i}, Y^2=1, YXY^{-1}=\zeta_{2^i}^{-1}X]$ have dimension 16 over their centers $E_i = Q(\zeta_d+\zeta_d^{-1}, \zeta_{2^i}+\zeta_{2^i}^{-1})$. The types are Sp for S_{k-2}; O for $S_{k-3},\ldots,S_2$; U for $M_2(Q(\zeta_d))$ and O for the last two factors.

Proof. This is similar to the previous results, but the factor $M_2(Q(\zeta_d))$ deserves some comments. In the natural decomposition of $S(d)$ it appears as

$$S_1 = Q(\zeta_d)^t[X,Y|X^2 = -1, Y^2 = 1, YXY^{-1} = -X],$$

and we define an explicit isomorphism to $M_2(Q(\zeta_d))$ as follows. Replace $\zeta_d-\zeta_d^{-1} \in Q(\zeta_d)$ by $\bar{Y} = (\zeta_d-\zeta_d^{-1})Y$. This is possible because $\bar{Y}^2 = (\zeta_d-\zeta_d^{-1})^2$. Since $X\bar{Y} = \bar{Y}X$, $Q(\zeta_d+\zeta_d^{-1})(\bar{Y})$ is central. But $Q(\zeta_d+\zeta_d^{-1})(\bar{Y}) \cong Q(\zeta_d)$ and

$$S_1 = Q(\zeta_d+\zeta_d^{-1})(\bar{Y})^t[X,Y] = (Q(\zeta_{4d})/Q(\zeta_d), Y, 1).$$

Since 1 is a norm, $S_1 = M_2(Q(\zeta_d))$ as claimed. //

In this paper we shall make complete calculations of L-theory only for periodic groups G whose 2-hyperelementary subgroups are of type I and type IIA. These are the cases where $\mathbb{Z}_2[\tau]$ is an Azumaya algebra. Let $O(G) < G$ be the maximal odd order normal subgroup. Then $G/O(G)$ belongs to one of the types

I. $\mathbb{Z}/2^k$, II. $Q2^k$, III. $S\ell_2(\mathbb{F}_3)$,

IV. $T\ell_2(\mathbb{F}_3)$, V. $S\ell_2(\mathbb{F}_p), p \geq 5$, VI. $T\ell_2(\mathbb{F}_p), p \geq 5$.

($T\ell_2(\mathbb{F}_p)$ is the degree 2 extension of $S\ell_2(\mathbb{F}_p)$ given by

1.18 $$T\ell_2(\mathbb{F}_p) = \langle S\ell_2(\mathbb{F}_p), S \mid SAS^{-1} = \theta(A), S^2 = -I\rangle$$

where θ represents the non-trivial element of $\mathrm{Out}(S\ell_2(\mathbb{F}_p))$, e.g. θ is conjugation with $\begin{pmatrix} 0 & 1 \\ -w & 0 \end{pmatrix}$, $w \in \mathbb{F}_p^\times - \mathbb{F}_p^{\times 2}$).
From [TW] one gets

Proposition 1.19. Let G be a group with periodic cohomology. The 2-hyperelementary subgroups of G have type I, IIA if and only if

(i) G has type I, III or V, or

(ii) G is an extension $0 \to \mathbb{Z}/n \to G \to \mathbb{Z}/m \times Q2^k \to 1$ with m and n odd coprime numbers and $\mathbb{Z}/n \tilde{\times} Q2^k$ of type IIA, or

(iii) $G = K \times S\ell_2(\mathbb{F}_p)$ or $G = K \times T\ell_2(\mathbb{F}_p)$ where $p \geq 5$ and $|K|$ is odd.

§2. Review of L-theory.

This paragraph reviews the calculational techniques of L-theory as developed by Wall in the papers [W1]-[W6].

Fix an antistructure (R,α,u). For $A \in G\ell(R)$, let A^* denote the transpose conjugate, $A^* = (A^\alpha)^t$. This defines an involution T on $K_1(R)$ and for each involutive subgroup $U \subset K_1(R)$ we let $H^i(U)$ denote the Tate cohomology groups,

$$H^i(U) = \mathrm{Ker}(1-(-1)^i T)/\mathrm{Im}(1+(-1)^i T).$$

In [W1], Wall associates to each involutive subgroup $U \subset K_1(R)$ groups $L_i^U(R,\alpha,u)$ such that $L_{i+2}^U(R,\alpha,u) = L_i^U(R,\alpha,-u)$.

Elements in $L_0^U(R,\alpha,u)$ are represented by non-singular quadratic forms on even dimensional free (right) R-modules, that is, by pairs (R^{2n},Q) where $Q \in M_{2n}(R)$, $B = Q + uQ^* \in G\ell_{2n}(R)$ and where the discriminant $\det((-u)^{-n}B)$ belongs to U. (The associated quadratic form on R^{2n} is $q(x) = x^*Qx$ and the bilinear form is $b(x,y) = x^*By$.)

Elements in $L_1^U(R,\alpha,u)$ are represented by automorphisms A of the hyperbolic form $H_\alpha(R^n) = \left(R^{2n}, \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\right)$ with $\det(A) \in U$.

For varying superscripts the L-groups are related by the Rothenberg exact sequence.

Theorem 2.1. ([W1]). Suppose $U \subset V$ are involutive subgroups of $K_1(R)$. There is an exact sequence

$$\cdots \to H^{i+1}(V/U) \xrightarrow{t_i} L_i^U(R,\alpha,u) \to L_i^V(R,\alpha,u) \xrightarrow{d_i} H^i(V/U) \to \cdots$$

We describe the homomorphisms t_i, d_i for $i = 0,1$; the other cases follow from $L_i(R,\alpha,u) = L_{i+2}(R,\alpha,-u)$. Let $A \in G\ell_n(R)$ represent $a \in V \subseteq K_1(R)$. Then

$$t_0(a) = \left(R^n \oplus R^n, \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix}\right), \quad t_1(a) = \begin{pmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{pmatrix}$$

If (R^{2n},Q) represents an element of $L_0^V(R,\alpha,u)$ then

$$d_0([R^{2n},Q]) = \det(-u)^{-n}\det(Q+uQ^*) \in H^0(V/U).$$

Finally, if A is an automorphism of $\left(R^{2n}, \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\right)$, then

$$d_1([A]) = \det(A) \in H^1(V/U).$$

We fix some notation.

2.2

E = finite Galois extension of Q

A = the algebraic integers in E

$P(E)$ = the set of all primes in E

$P_n(E)$ = the set of primes which divides n, $(n \leq \infty)$

E_y, A_y = completions at $y \in P(E)$; $E_n = \prod_{y|n} E_y$

Let S be a finite dimensional simple Q-algebra with center E. Consider finite sets Ω, $P_\infty(E) \subset \Omega \subset P(E)$, with the property that $S_y = E_y \otimes_E S$ is split (i.e. $S_y \cong M_m(E_y)$) if $y \notin \Omega$. Let

2.3

$$S_\Omega = \prod\{S_y \mid y \in \Omega\} \times \prod\{M_m(A_y) \mid y \notin \Omega\}$$

$$S_A = \varinjlim S_\Omega, \quad \hat{S} = S_A/\prod\{S_y \mid y \in P_\infty(E)\}$$

(alternatively, $S_A = Q_A \otimes S$ and $\hat{S} = \hat{Q} \otimes S$). Similar notation is used for semi-simple algebras.

Let $R \subset S$ be a $\mathbb{Z}$-order. We have $\hat{R} \subset \hat{S}$ and $\hat{R} = \prod R_p$, $R_p = \prod\{R_y \mid y \in P_p(E)\}$. Here are some useful facts about K_1-groups (see [B] and [W3]). The reduced norm gives isomorphisms

2.4

$$\mathrm{Nrd}: K_1(S_y) \xrightarrow{\cong} C(S_y)^\times, \quad y \in P(E) - P_\infty(E)$$

$$\mathrm{Nrd}: K_1(\hat{S}) \xrightarrow{\cong} C(\hat{S})^\times, \quad \mathrm{Nrd}: K_1(S) \xrightarrow{\cong} C(S)^\bullet$$

where $C(S)^* \subset C(S)^\times$ is the subset of elements which become positive at all $y \in P_\infty(E)$ where $E_y = \mathbb{R}$ and S_y is non-split ($S_y \cong M_n(\mathbb{H})$). Here $C(\ )$ denotes the center.

Let $J_y \subseteq R_y$ be an ideal such that $R_y = \varprojlim(R_y/J_y^N)$. There is an exact sequence

2.5 $$1 + J_y \to K_1(R_y) \to K_1(R_y/J_y) \to 0$$

Write $X(R) = \mathrm{Ker}\{K_1(R) \to K_1(S)\}$, $X(R_y) = \mathrm{Ker}\{K_1(R_y) \to K_1(S_y)\}$, and $K_1' = K_1/X$. We have the exact sequence

2.6 $$0 \to K_1'(R) \to K_1'(\hat{R}) \oplus K_1(S) \to K_1(\hat{S})$$

whose terms are all calculable via reduced norms.

Note, if $S = (L/E, f)$ is a crossed-product algebra, then $S^\times \xrightarrow{i} K_1(S) \xrightarrow{\mathrm{Nrd}} C(S)^\times$ is given by

2.7 $$\mathrm{Nrd}(i(s)) = \det(\lambda_s)$$

where $\lambda_s \in M_n(L)$ is the matrix (w.r.t. the basis $\{u_\sigma\}$) of right multiplication of s on $S = \Sigma L u_\sigma$, cf. 1.8.

Consider a 2-hyperelementary group $\pi = \mathbb{Z}/n \tilde{\times} \sigma$ specified by $t\colon \sigma \to (\mathbb{Z}/n)^\times$. Its rational group ring decomposes into twisted group rings

$$\mathbb{Q}[\mathbb{Z}/n \tilde{\times} \sigma] = \prod_{d|n} \mathbb{Q}(\zeta_d)^t[\sigma]$$

with twisting $t\colon \sigma \to (\mathbb{Z}/d)^\times$. Each factor can be further decomposed into simple rings which are crossed-product algebras. For the groups in 1.1 this was carried out explicitly at the end of §1. Define

$$R(d) = \mathbb{Z}[\zeta_d]^t[\sigma], \quad S(d) = \mathbb{Q}(\zeta_d)^t[\sigma].$$

The rings have anti-involutions, denoted by α, inherited from the group rings. Write $L_n^X(R)$ instead of $L_n^X(R,\alpha,1)$. From [W5], [W6] we have

<u>Theorem 2.8</u>. (i) There is a natural decomposition

$$L_k^X(\mathbb{Z}\pi) = \prod_{d \nmid n} L_k^X(\mathbb{Z}\pi)(d)$$

(ii) Each factor in (i) fits into an exact sequence

$$\cdots \to L_{k+1}^X(\hat{S}(d)) \to L_k^X(\mathbb{Z}\pi)(d) \to \prod_{p \nmid d} L_k^X(R_p(d)) \times L_k^X(\hat{S}(d)) \to \cdots$$

where $R_p(d) = \mathbb{Z}_p \otimes R(d)$.

We now review how to calculate the individual terms in 2.8(ii). Let $S \subset S(d)$ be a simple factor with center E. If $\alpha|E \neq 1_E$ (type U) then α acts on $P_p(E)$ and

$$S_p = \Pi\{S_y \mid y \in P(E)^\alpha\} \times \Pi\{S_y \times S_{\alpha(y)} \mid y \in P_p(E)^\alpha\}.$$

where the first factors have type U; the last have type GL. If $p \nmid 2d$ then $S_p = M_m(E_p)$, and $M_m(A_p)$ is a factor in $R_p(d)$, so we have

2.9 $$S_p(d) = \Pi M_{r_i}(E_{i,p}), \quad R_p(d) = \Pi M_{r_i}(A_{i,p}) \quad (p \nmid 2d)$$

<u>Proposition 2.10</u>. ([W0]). For type GL antistructures, $L_i^V(R,\alpha,u) = 0$.

<u>Proposition 2.11</u>. ([W3]). Let R be a complete p-local ring and $J \subseteq \mathrm{rad}(R)$ a (2-sided) ideal. Reduction gives an isomorphism

$$L_i^K(R,\alpha,u) \xrightarrow{\cong} L_i^K(\bar{R},\bar{\alpha},\bar{u})$$

where $\bar{R} = R/J$. If $p \neq 2$ reduction modulo J also gives isomorphism of $L_i^{\{0\}}$-groups.

Proposition 2.12. Morita equivalent antistructures have the same L_*^V groups for any V.

This is direct from the definition 1.7 and can be outlined as follows. First, suppose (R,α,u) and (R,β,v) are scaling equivalent in the sense of 1.3. Let (R^{2n},Q) represent an element of $L_{2n}(R,\alpha,u)$. The associated element of $L_{2n}(R,\beta,v)$ is (R^{2n},rQ). Second, a form (R^{2nr},Q) over (R,α,u) gives the form $(M_r(R)^{2n},Q)$ over $(M_r(R),\alpha_r,uI_r)$ and vice-versa. Since forms based on R^{2nr} are cofinal in all forms considered, this shows that L_{2n}-groups are unchanged under Morita equivalence. Similar considerations apply to L_{2n+1}.

Proposition 2.13. ([W5]). In the setting of 2.3,

$$L_n^V(S_A,\alpha,u) = \varinjlim L_n^V(S_\Omega,\alpha,u)$$

and $L_n^V(S_\Omega,\alpha,u) = \Pi L_n^V(S_y,\alpha,u) \times L_n^V(M_r(A_y),\alpha,u)$.

The results up to this point in principle reduces the calculation of 2.8(ii) to $L_*(D,\alpha,u)$ for division algebras. In stating the results in these cases we shall often use expressions like $H^*(K_1(D))$, $H^x(E^x)$ etc. This means Tate cohomology with respect to the involution induced from the antistructure in question. For example, if (E,α,u) has type O or type Sp then $H^0(E^x) = E^x/(E^x)^2$ and $H^1(E^x) = \langle\pm 1\rangle$.

Proposition 2.14. ([W1]). Let D be a division algebra with center E. Then $L_1^K(D,\alpha,u) = 0$ except if $(D,\alpha,u) = (E,1,1)$ where $L_1^K(E,1,1) = \mathbb{Z}/2$.

For every antistructure, $L_1^K(R,\alpha,u)$ contains the element $\langle\tau\rangle$ represented by the matrix $\begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \in Gl_2(R)$. The non-trivial element $L_1^K(E,1,1)$ is represented by τ in all cases. Note that τ lifts to $L_1^V(R,\alpha,u)$ precisely when $[-u] = 0$ in $H^1(K_1(R)/V)$; and then $2\langle\tau\rangle = \langle\tau^2\rangle = \langle\begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix}\rangle = t_1([u])$. Thus if also $[u] \neq 0$ in $H^0(K_1(R)/V)$ then $\langle\tau\rangle \in L_1^V(R,\alpha,u)$ has order 4.

The even L-groups $L_{2n}^V(D,\alpha,u)$ for antistructures over division algebras depend on the structure of the center E. We list some relevant results for $V = \{0\}$ (where we write L_* instead of $L_*^{\{0\}}$).

We shall give the group together with generators/detecting invariants. We use the following notation and invariants.

(i) $t_n: H^{n+1}(K_1(D)) \to L_n(D,\alpha,u)$ from 2.1. When t_n is bijective, its inverse t_n^{-1} is the Paffian.

(ii) $\sigma: L_{2n}(D,\alpha,u) \to \mathbb{Z}$ $(E = \mathbb{C}$ or $\mathbb{R})$ is the signature homomorphism.

(iii) $\langle c\rangle = \left[(\mathbb{F}_q^2, \begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix})\right] \in L_0(\mathbb{F}_q,1,1)$, $q = 2^i$. The class $\langle c\rangle$ generates a $\mathbb{Z}/2$ precisely when $b \in \mathbb{F}_q$ is not of the form $x + x^2$. This is the Arf invariant element.

Let E be a local field with prime element π, and let L/E the unique quadratic unramified extension. The reduced norm Nrd: $C_2(E) \to E$ of the cyclic algebra $C_2(E) = (L/E,X,\pi)$ gives a quadratic form on E^4, which determines a class of order 2

(iv) $\langle\kappa\rangle = [C_2(E), \mathrm{Nrd}] \in L_0(E,1,1)$ (E local)

This is the Hasse invariant element.

The following tables are contained in [W2-4]. They list the value of $L_n(D) = L_n^{\{0\}}(D,\alpha,u)$ for division rings D according to the type of (D,α,u) (when the center $E = C(D)$ is finite, local or continuous). Since $L_n(D,\alpha,u) = L_{n+2}(D,\alpha,-u)$ and the types O and Sp are interchanged when u is replaced by $-u$ it suffices to list type O and type U.

Table 2.15: finite fields.

(i) type U: $L_n(\mathbb{F}_q) = 0$

(ii) type O, q odd: $L_0(\mathbb{F}_q) = 0$; $t_1^{-1}: L_1(\mathbb{F}_q) \cong H^0(\mathbb{F}_q^\times)$;

$t_2^{-1}: L_2(\mathbb{F}_q) \cong H^1(\mathbb{F}_q^\times)$; $L_3(\mathbb{F}_q) = 0$.

(iii) type O, $q = 2^i$: $L_{2n}(\mathbb{F}_q) = \mathbb{Z}/2\langle c\rangle$; $L_{2n+1}(\mathbb{F}_q) = \mathbb{Z}/2\langle\tau\rangle$.

(Note from 2.1 that $L_n^K(\mathbb{F}_{2^i}) \cong L_n(\mathbb{F}_{2^i})$ since $H^*(\mathbb{F}_{2^i}^\times) = 0$)

Table 2.16: E local field

(i) type U: $L_n(D) = 0$

(ii) type O: $t_0^{-1}: L_0(D) \cong H^1(E^\times)$ if $D \neq E$ and

$L_0(E) \cong \mathbb{Z}/2\langle\kappa\rangle$; $t_1^{-1}: L_1(D) \cong H^0(E^\times)$; $t_2^{-1}: L_2(D) \cong H^1(E^\times)$;

$L_3(D) = 0$.

Table 2.17: $E = \mathbb{R}$ or $E = \mathbb{C}$.

(i) type U: $L_{2n+1}(D) = 0$, $\sigma: L_{2n}(D) \cong 4\mathbb{Z}$

(ii) type O:

(a) $D = \mathbb{C}$: $L_0(D) = 0$; $L_1(D) = 0$; $t_2^{-1}: L_2(D) \cong H^1(E^\times)$;
$L_3(D) = 0$

(b) $D = \mathbb{R}$: $\sigma: L_0(D) \cong 4\mathbb{Z}$, $t_1^{-1}: L_1(D) \cong H^0(E^\times)$,
$t_2^{-1}: L_2(D) \cong H^1(E^\times)$, $L_3(D) = 0$

(c) $D = \mathbb{H}$: $L_0(D) = 0$, $L_1(D) = 0$, $\sigma: L_2(D) \cong 2\mathbb{Z}$,
$L_3(D) = 0$.

Let D be a division algebra over the global field E. There is an exact sequence

2.18 $$0 \to L_n(D,\alpha,u) \to L_n(D_A,\alpha,u) \overset{p}{\to} CL_n(D,\alpha,u) \to 0$$

where D_A is the adèle ring (cf.2.3) and

Table 2.19: E global field.

(i) type U: $CL_n = 0$

(ii) type O: $CL_n = \mathbb{Z}/2$, $H^0(C(E))$; $H^1(C(E))$; 0 for $n = 0,1,2,3$

Here $C(E) = E_A^\times/E^\times$ is the idèle class group of the center. The information above about the odd L-groups can all be derived quite easily from 2.14. Also note by 2.12 that the results for D extend to results for simple (finite dimensional) E-algebras. Finally, we need to know how the tables 2.16 and 2.17 are mapped to 2.19.

Addendum 2.20. For (D,α,u) of type O and each prime $y \in P(E)$, $p: L_n(D_y) \to CL_n(D)$ is the obvious map when $y < \infty$ and also on the torsion part if $y = \infty$. Moreover, each summand $4\mathbb{Z}$ surjects onto $\mathbb{Z}/2$ for $n=0$. If $n=2$ and $D_y = \mathbb{H}$ then $2\mathbb{Z}$ surjects onto the corresponding $\langle \pm 1\rangle$ in $H^1(E_\infty^\times)$.

We can add in 2.8(ii) the term $L_n(T(d))$ to both $\Pi L_n(R_p(d))$ and to $L_n(\hat{S}(d))$ and then use 2.18, 2.19 to get the "*main exact sequence*" from [W5], [W6],

2.21 $\ldots \xrightarrow{\psi_{n+1}} CL_{n+1}(S(d)) \to L_n^X(\mathbb{Z}\pi)(d) \to L_n(T(d)) \times \prod_{p \nmid d} L_n^X(R_p(d)) \to \ldots$

where the suppressed antistructure is $(R,\alpha,1)$ in all cases. We close by noting that the restriction γ_n of ψ_n,

2.22 $$\gamma_n: L_n(T(d)) \times \prod_{p \nmid 2d} L_n^X(R_p(d)) \to CL_n(S(d))$$

decomposes according to the simple components of $(S(d),\alpha,1)$; Thus it is convenient first to calculate $\operatorname{Ker}\gamma_n$ and $\operatorname{Coker}\gamma_n$ and then to use the exact sequences below to give $L_n^X(\mathbb{Z}\pi)(d)$,

2.23
$$0 \to \operatorname{Coker}\psi_{n+1} \to L_n^X(\mathbb{Z}\pi)(d) \to \operatorname{Ker}\psi_n \to 0$$
$$0 \to \operatorname{Ker}\gamma_n \to \operatorname{Ker}\psi_n \to L_n^X(R_2(d)) \xrightarrow{\bar{\psi}_n} \operatorname{Coker}\gamma_n \to \operatorname{Coker}\psi_n \to 0.$$

Details can be found in §5.

§3. Calculation of $L_n^X(R_2(d))$.

We consider a 2-hyperelementary group $\pi = \rho \tilde{\times} \sigma$, where ρ is a normal cyclic subgroup of odd order n and σ is a 2-group. Then π is specified by $t: \sigma \to \rho^\times$. For each divisor d of n consider $t: \sigma \to \rho^\times \to (\mathbb{Z}/d)^\times$. Throughout the paragraph we let

3.1 $$\sigma_2 = \operatorname{Ker}(t: \sigma \to (\mathbb{Z}/d)^\times), \quad \sigma_1 = t^{-1}(\langle 2\rangle)$$

where $\langle 2\rangle \subset (\mathbb{Z}/d)^\times$ is the group consisting of powers of 2.

The 2-local isomorphism $\mathbb{Z}_{(2)}[\rho] = \prod_{d|n} \mathbb{Z}_{(2)}[\zeta_d]$ gives a splitting (of antistructures)

$$\mathbb{Z}_2\pi = \prod_{d|n} R_2(d), \quad R_2(d) = \mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]^t\sigma$$

and $L_1^X(\mathbb{Z}_2\pi) = \Pi L_1^X(R_2(d))$; as usual we suppress the anti-involution α and the unit $u = 1$. We fix a divisor $d > 1$ of n.

The ring $\mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]$ is a direct sum of isomorphic copies of $B = \mathbb{Z}_2[\zeta_d]$, one for each 2-adic prime y in $\mathbb{Z}[\zeta_d]$. The Galois group $\mathbb{Z}/d^\times$ of $Q(\zeta_d)$ acts on it permuting the summands B transitively and the stabilizer of each B is equal to its Galois group $\langle 2\rangle \subset \mathbb{Z}/d^\times$. Thus σ/σ_1 permutes the summands B freely and σ_1/σ_2 is a subgroup of $\operatorname{Gal}(B/\mathbb{Z}_2)$.

If σ/σ_1 has g_2 orbits of size $r = (\sigma:\sigma_1)$ in the set of all 2-adic primes $P_2(Q(\zeta_d))$, then $R_2(d)$ is a direct sum of g_2 copies of $(\prod_{\sigma/\sigma_1} B)^t\sigma$. Each copy is a crossed product algebra of $(B^t\sigma_1)^r$ by σ/σ_1; and is isomorphic to $M_r(B^t\sigma_1)$ (cf. [W7, 3.3] and 3.2 below). The center of $R_2(d)$ contains the subring $A_2 = \mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]^\sigma$, which is a sum of g_2 copies of the 2-ring $A = B^{\sigma_1}$, Let $\operatorname{rad}(R)$ be the Jacobson radical of the ring R and set $\bar{R} = R/\operatorname{rad}(R)$; then $\bar{A}_2 = g_2\bar{A}$ where $\bar{A}$ is a finite field of characteristic 2. Note that g_2 is the index of the subgroup $\sigma/\sigma_2 \cdot \langle 2\rangle$ in $\mathbb{Z}/d^{\times}$. In the lemma below, type O (resp. type U) means that α is trivial (resp. non-trivial) on the center.

<u>Lemma 3.2</u>. We have isomorphisms $\overline{R_2(d)} \cong M_q(\bar{A}_2) \cong g_2 M_q(\bar{A})$. where $q = (\sigma:\sigma_2)$. The antistructure $(\overline{R_2(d)},\alpha,1)$ has

type GL if $-1 \notin (\sigma/\sigma_2 \cdot \langle 2\rangle)$

type U if $-1 \in (\sigma/\sigma_2 \cdot \langle 2\rangle) - \sigma/\sigma_2$

type O if $-1 \in \sigma/\sigma_2$

<u>Proof</u>. By the above $R_2(d) \cong g_2 M_r(B^t[\sigma_1])$ and there are isomorphisms

$$\overline{B^t[\sigma_1]} \cong \overline{B^t[\sigma_1/\sigma_2]} \cong \overline{\bar{B}^t[\sigma_1/\sigma_2]}$$

induced from the natural projections. As B is unramified over $\mathbb{Z}_2$, $\mathrm{Gal}(B/\mathbb{Z}_2) = \mathrm{Gal}(\bar{B}/\mathbb{F}_2)$ and the subgroup σ_1/σ_2 is the Galois group of $\bar{A} = \bar{B}^{\sigma_1/\sigma_2} \subset \bar{B}$. Thus $\bar{B}^t[\sigma_1/\sigma_2]$ is isomorphic to the cyclic algebra $(\bar{B}/\bar{A},X,1) \cong M_s(A)$, $s = (\sigma_1:\sigma_2)$, and hence

$$\overline{R_2(d)} \cong g_2\, M_r(M_s(\bar{A})) \cong M_q(\bar{A}_2).$$

The involution α corresponds to -1 in the Galois group $(\mathbb{Z}/d)^{\times}/(\sigma/\sigma_2)$ of $\mathbb{Z}[\zeta_d]^{\sigma}$ over $\mathbb{Z}$, and the action on the 2-adic completion A_2 is faithfully described on the residue level $\bar{A}_2$. As the Galois group of one summand $\bar{A}$ over $\mathbb{F}_2$ is the subgroup $(\sigma/\sigma_2 \cdot \langle 2\rangle)/(\sigma/\sigma_2)$ this gives the claim. //

Next we study the involution on $R_2(d) \cong g_2 M_r(B^t\sigma_1)$. If $\overline{R_2(d)}$ has type GL, so does $R_2(d)$ since the components of

$A_2 \cong g_2 A$ are interchanged in pairs and each summand $M_r(B^t\sigma_1)$ contains one of them. Suppose then that $\overline{R_2(d)}$ has type U or O, so that each summand $M_r(B^t\sigma_1)$ is invariant under α. Choose one component, say B in $\mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]$. If $g_0 \in \sigma$ has image -1 in $(\mathbb{Z}/d)^{\times}/\langle 2\rangle$, then α maps B to $g_0 B g_0^{-1}$. To keep B invariant we scale α to $\beta = g_0^{-1}\alpha g_0$, so that $(M_r(B^t\sigma_1),\alpha,1) \sim (M_r(B^t\sigma_1),\beta,g_0^2)$ (the elements g_0^{-2} and g_0^2 have the same class in $H^1(K_1(B^t\sigma_1))$.

<u>Lemma 3.3</u>. Let $g_0 \in \sigma$ have image -1 in $\mathbb{Z}/d^{\times}/\langle 2\rangle$ and let $\beta = g_0^{-1}\alpha g_0$. Morita equivalence gives an isomorphism of involutive groups $K_1(B^t\sigma_1) \cong K_1(M_r(B^t\sigma_1))$ where the involutions on both sides are induced from β.

<u>Proof</u>. Let $S = \{1,g_2,\dots,g_r\} \subset \sigma$ be a set representing σ/σ_1. Let B_i be the component of $\mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]$ corresponding to $g_i(B_1 = B)$. Then $R = (\Pi B_i)^t\sigma$ is a component of $R_2(d)$ isomorphic to $M_r(R_1)$, $R_1 = B^t\sigma_1$, and we must make the isomorphism explicit. Let $V = \oplus\{gR_1 \mid g \in S\}$. It is right R_1-module and the given basis induces as isomorphism $\mathrm{End}_{R_1}(V) \cong M_r(R_1)$. On the other hand V is also a left ideal of R and $\varphi\colon R \to \mathrm{End}_{R_1}(V) \cong M_r(R_1)$ given by $\varphi(r)(v) = rv$, is an isomorphism. Let $e_1 \in R_1$ be the 1-element and write $e_i = g_i e_1^{-1} g_i \in R$. It follows easily that $\varphi(r_1e_1+e_2+\dots+e_r)$ is the diagonal matrix $\mathrm{diag}(r_1,1,\dots,1)$. Hence there is a commutative diagram

$$\begin{array}{ccc} R_1^{\times} & \xrightarrow{\ i\ } & R^{\times} \\ \downarrow & & \downarrow \\ K_1(R_1) & \xrightarrow{\text{Morita}} & K_1(R^{\times}) \end{array}$$

where $i(r_1) = r_1e_1 + e_2+\dots+e_r$. The vertical arrows are surjections and i is clearly involutive (w.r.t. β). The lemma follows. //

Remark. We do not know in general if the anti-involution β on $M_r(B^t\sigma_1)$ in 3.2 is scaling equivalent to the conjugate transpose involution $\beta_r((m_{ij})) = (\beta(m_{ji}))$. However, if σ/σ_2 is cyclic or if σ_2 as abelian then β is actually equivalent to β_r and there is a Morita equivalence of antistructures $(M_r(B^t\sigma_1),\beta,g_0^2) \sim (B^t\sigma_1,\beta,g_0^2)$. One would expect this to hold in general.

Theorem 3.4. Let $\tau = \mathbb{Z}/n \tilde{\times} \sigma$ be a 2-hyperelementary group. Then $L_i^X(R_2(d)) = 0$ for all d such that -1 is not in the image of σ in $\mathbb{Z}/d^\times$.

Proof. If $-1 \notin \sigma/\sigma_2$ then $\overline{R_2(d)}$ has type GL or type U by 3.2. In the first case also $R_2(d)$ has type GL and L_i^X vanishes by 2.10. Suppose then $\overline{R_2(d)}$ has type U. By 2.1 it suffices to show that $L_i^K(R_2(d)) = 0$ and $H^i(K_1'(R_2(d))) = 0$. From 2.11, 2.12 and 3.2

$$L_i^K(R_2(d),\alpha,1) \simeq L_i^K(\overline{R_2(d)},\alpha,1) \simeq L_i^K(\bar{A}_2,\alpha,1)$$

As $\bar{A}_2 = g_2\bar{A}$ is a sum of type U finite fields $\bar{A}$, $L_i^K(\bar{A}_2,\alpha,1) = 0$ by 2.15. Thus $L_i^K(R_2(d),\alpha,1) = 0$.

We use the involution $\beta = g_0^{-1}\alpha g_0$ where $g_0 \in \sigma$ has image -1 in $\mathbb{Z}/d^\times/\langle 2\rangle$ to calculate $H^i(K_1'(R_2(d)))$. By restriction β defines an involution on $B^t\sigma_1$ and from 3.3 $H^i(K_1'(R_2(d)))$ is g_2 copies of $H^i(K_1'(B^tg_1))$. Moreover, there is an isomorphism

$$K_1'(B^t\sigma_1) \simeq \check{H}^0(\sigma_1/\sigma_2;\ K_1'(B\sigma_2))$$

where $\check{H}^i$ denotes the usual cohomology groups, not the Tate cohomology groups ($\check{H}^0$ is the invariants). Ajoin an element $\tilde{g}_0$ to σ_1/σ_2 which satisfies $\tilde{g}_0^2 = g_0^2$ and $\tilde{g}_0^{-1}h\tilde{g}_0 = \beta(h) = g_0^{-1}h^{-1}g_0$. If $t(g_0) = (-1)\cdot 2^k$ in $(\mathbb{Z}/d)^\times$, we let $\tilde{g}_0^{-1}\zeta_d\tilde{g}_0 = \zeta_d^{2^k}$. Now $\tilde{\sigma}_1/\sigma_2 = \langle\tilde{g}_0,\sigma_1/\sigma_2\rangle$ is cyclic as a subgroup of the Galois group $\langle 2\rangle$ of B over $\mathbb{Z}_2$, and $2^k \notin \sigma_1/\sigma_2$ since otherwise one would have $-1 \in \sigma_1/\sigma_2$ contradicting the assumption of type U. Hence $\tilde{g}_0$ generates $\tilde{\sigma}_1/\sigma_2$ and we have a compatible action on $K_1'(B\sigma_2)$.

The proof of [O3, Lemma 5] shows that $K_1'(B\sigma_2)$ is cohomologically trivial both with respect to σ_1/σ_2 and $\tilde{\sigma}_1/\sigma_2$. The spectral sequence of the extension $0 \to \sigma_1/\sigma_2 \to \tilde{\sigma}_1/\sigma_2 \to \mathbb{Z}/2 \to 0$ and periodicity imply

$$H^i(\mathbb{Z}/2;K_1'(B^t\sigma_1)) = \check{H}^{i+2}(\mathbb{Z}/2;\check{H}^0(\sigma_1/\sigma_2;K_1'(B\sigma_2)))$$

$$\simeq \check{H}^{i+2}(\tilde{\sigma}_1/\sigma_2;K_1'(B\sigma_2)) = 0$$

for $i = 0$ and 1. //

Corollary 3.5. If τ is 2-hyperelementary and ρ is cyclic of odd order, then

$$L_i^X(\mathbb{Z}_2[\tau\times\rho]) \simeq L_i^X(\mathbb{Z}_2\tau).$$

Proof. If $\tau = \mathbb{Z}/n \tilde{\times} \sigma$, σ a 2-group and $\rho = \mathbb{Z}/m$, then

$$L_i^X(\mathbb{Z}_2[\tau\times\rho]) = \oplus L_i^X(R_2(ad))$$

summed over (a,d) with $a \in m$ and $d|n$. If $a > 1$, then $-1 \neq 1$ in $(\mathbb{Z}/a)^\times$ and since σ maps trivially into $(\mathbb{Z}/a)^\times$, $L_i^X(R_2(ad)) = 0$ by 3.4. Thus we get only the sum over $d|n$, which is equal to $L_i^X(\mathbb{Z}_1\tau)$. //

We can globalize 3.5 to integral group rings, generalizing

[W6, Theorem 2,4,2]. Indeed, using induction ([D]) and the main exact sequence 2.21 we have

<u>Theorem 3.6</u>. Let π be an arbitrary finite group and let ρ be a group of odd order. Then

$$L_i^X(\mathbb{Z}[\pi\times\rho]) \cong L_i^X(\mathbb{Z}\pi)\oplus\Sigma$$

where $\Sigma = \mathrm{Ker}(L_i(\mathbb{R}[\pi\times\rho]) \to L_i(\mathbb{R}\pi))$ is free abelian.

Note that 3.6 is false for the simple surgery obstruction groups $L_i^s(\pi) = L_i^U(\mathbb{Z}\pi)$ where $U = \{\pm 1\}\oplus\pi/\pi'$. This follows from 3.6 and calculations from [O2]: $SK_1(\mathbb{Z}[Q2^k]) = 0$ but $SK_1(\mathbb{Z}[\mathbb{Z}/p\times Q2^k]) = \mathbb{Z}/2$ when p is an odd prime.

In the rest of §3 we make the

<u>Assumptions</u>. $-1\in \mathrm{Image}(t\colon \sigma\to(\mathbb{Z}/d)^\times)$, σ_2 is abelian.

In these cases, $R_2(d)\cong g_2\cdot M_r(B^t\sigma_1)$ is actually a matrix ring over its center. First recall

<u>Lemma 3.7</u>. Let C be a complete local ring with maximal ideal m and let R be a C-algebra which is free as a C-module. If $R/mR\cong M_n(C/m)$, then $R\cong M_n(C)$.

<u>Proof</u>. Let $\bar C = C/m$ and $\bar R = \bar C\otimes R = R/mR$. Choose a minimal left ideal $\bar I$ in $\bar R$. It has the form $\bar I = \bar R\bar e$ with $\bar e^2 = \bar e$ (e.g. $\bar e = \bar e_{11}+\bar e_{21}+\dots+\bar e_{n1}$, $\bar e_{ij}\in M_n(\bar C)$ the standard matrix unit). Then $\bar I\cong\bar C^n$ and the isomorphism $\bar R\cong\bar M_n(C)$ can be realized as

$$\bar f\colon \bar R\to \mathrm{Hom}_{\bar C}(\bar I,\bar I)$$

where $\bar f(\bar r)$ is left multiplication by $\bar r$.

The idempotent $\bar e\in\bar R$ can be lifted to an idempotent e of R since C is complete. As a direct summand of the free C-module R, the left ideal $I = Re$ is projective, hence free over the local ring C. By Nahayama's lemma any $\bar C$-basis of $\bar I$ can be lifted to a C-basis of I, so that $I\cong C^n$ as a C-module. The analogously defined C-algebra homomorphism $f\colon R\to\mathrm{Hom}_C(I,I)$ has bijective reduction $\bar f$. As both sides are free C-modules, f is an isomorphism. //

Let $x\in\sigma_1$ represent a generator of σ_1/σ_2. It acts on $B\sigma_2$ by conjugation and the fixed subring C is the center of $B^t\sigma_1$. Moreover, $C_2 = g_2C$ is the center of $R_2(d) = g_2M_r(B^t\sigma_1)$.

<u>Proposition 3.8</u>. If σ_2 is abelian then $R_2(d)\cong M_q(C_2)$ where $q = (\sigma:\sigma_2)$.

<u>Proof</u>. Since $R_2(d)\cong g_2M_r(B^t\sigma_1)$, $r = (\sigma:\sigma_1)$, it suffices to show that $B^t\sigma_1\cong M_s(C)$, $s = (\sigma_1:\sigma_2)$. The augmentation $\varepsilon\colon C\to B^{\sigma_1} = A$ is a surjection of 2-adic orders and induces an isomorphism $\bar C\cong\bar A = \mathbb{F}$, so $m = \mathrm{rad}(C) = \varepsilon^{-1}(2A)$ is a maximal ideal and C is a local ring. We apply 3.7 to $R = B^t\sigma_1$ and (C,m).

First, R is a C-algebra since C is the center of R. We claim that mR is the radical of R. By 3.2, $\mathrm{rad}(R)$ is generated by 2 and the augmentation ideal $I(\sigma_2)$ of $B\sigma_2$. Clearly $2\in mR$ and we show $I(\sigma_2)\subset mR$ by induction on the order of σ_2. The subgroup ρ of σ_2 left fixed by x is non-trivial and $I(\rho) \subset mB$. If $\sigma_2' = \sigma_2/\rho$ we thus have $B\sigma_2/(m) \cong B\sigma_2'/(m')$, isomorphic to $\bar{\bar{B}}$ by induction. Hence $mR = rad(R)$ and $R/mR = \bar{R} = M_s(\mathbb{F})$.

We still have to check that R is free over C. Since $\bar{R} = M_s(\mathbb{F})$, Nahayama's lemma shows that R is generated by s^2 elements over C. By counting the ranks over A we see that there can be no relations. Indeed, $R = B^t\sigma_1$ is free of rank $s|\sigma_1|$ over A, and C is generated additively by $(b,g) = \sum_{i=0}^{\ell} b^{x^i} g^{x^i}$ where $g \in \sigma_2$, $\ell > 0$ is minimal with respect to $g^{x^\ell} = g$, and $b \in B$ is fixed under x^ℓ. Now $x^s \in \sigma_2$ so $\ell | s$ for all g, and it follows that C is free of rank $|\sigma_2|$ over A. Since $s^2|\sigma_2| = s|\sigma_1|$ this proves the claim. //

The center C_2 is the fixed subring of $(\mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d])\sigma_2$ under σ/σ_2. Let B_1 be one component of $\mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]$, let $S \subset \sigma$ represent σ/σ_1, and denote $B_g = gB_1g^{-1}$ for $g \in S$. Then the component of $R_2(d)$ corresponding to B_1 is $(\Pi B_g)^t[\sigma] \cong M_r(B_1^t\sigma_1)$ and its center is

$$C = \{(r^g)_{g\in S} \mid r \in (B_1\sigma_2)^{\sigma_1}\} \subset \Pi(B_g\sigma_2)$$

The involution α acts on C by

$$\alpha(r^g)_{g\in S} = (\alpha(r)^g)_{g\in S} = ((g_0^{-1}\alpha(r)g_0)^{gg_0})_{g\in S}$$

where $g_0 \in \sigma$ has image -1 in $\mathbb{Z}/d^\times/\langle 2\rangle$. When g runs through S, so does gg_0 modulo elements in σ_1. But r is invariant under σ_1 and $\beta(r) = g_0^{-1}\alpha(r)g_0 \in (B_1\sigma_2)^{\sigma_1}$, so $\alpha|_C = \beta|_{(B_1\sigma_2)^{\sigma_1}}$.

There are now two possibilities. If $-1 \in \langle 2\rangle$ then we can choose $g_0 = 1$ and the involution is the usual one in $(B_1\sigma_2)^{\sigma_1}$. If $-1 \notin \langle 2\rangle$ so that the extension $\mathbb{Z}[\zeta_d+\zeta_d^{-1}] \subset \mathbb{Z}[\zeta_d]$ is split at the dyadic primes, we choose $g_0 \in \sigma$ with image -1 in $\mathbb{Z}/d^\times$ (not only in $\mathbb{Z}/d^\times/\langle 2\rangle$); then $\beta = g_0^{-1}\alpha g_0$ satisfies $\beta(\zeta_d) = \zeta_d$ and $\beta(h) = g_0^{-1}h^{-1}g_0$, $h \in \sigma_2$.

<u>Proposition 3.9</u>. Suppose σ_2 is abelian. There is a Morita equivalence $(R_2(d),\alpha,1) \sim g_2\cdot(C,\beta,u)$, where $C = (\mathbb{Z}_2[\zeta_d]\sigma_2)^{\sigma_1}$ and $u \in C^\times$.

(i) If $-1 \in \langle 2\rangle$ then $\beta(\zeta_d) = \zeta_d^{-1}$, $\beta(h) = h^{-1}$ for $h \in \sigma_2$

(ii) If $-1 \notin \langle 2\rangle$ then $\beta(\zeta_d) = \zeta_d$, $\beta(h) = g_0^{-1}h^{-1}g_0$ for $h \in \sigma_2$ where $g_0 \in \sigma$ has image -1 in $(\mathbb{Z}/d)^\times$.

<u>Proof</u>. We saw above that the restriction of α to C is equal to β. Now $R_2(d) \cong g_2M_q(C)$ and each summand is an Azumaya algebra. Since C is a local ring, $Pic(C) = 0$ and $(M_q(C),\alpha,1) \simeq (M_q(C),\beta,u)$ for some $u \in C^\times$. //

We next determine the rational types of $S(d) = Q(\zeta_d)^t[\sigma]$. This gives information about the unit u in $H^1(C^\times)$.

Write $\tau = \mathbb{Z}/d \tilde{\times} \sigma$ so that $S(d)$ is a summand of $Q\tau$. It is semisimple and splits into simple summands $S(d) = \Pi S_i$. These are related to the complex characters of τ as follows.

If $\chi: \tau \to \mathbb{C}$ is the character of an irreducible complex representation $\rho: \mathbb{C}\tau \to M_d(\mathbb{C})$, injective on $\mathbb{Z}/d$, then $\rho(Q\tau)$ is isomorphic to a simple component S_i of $S(d)$. Conversely, each S_i has the form $M_n(D)$ where D is a skew field of dimension m^2 over its center K. The reduced norm gives a character $X: Q\tau \to M_n(D) \to K$; which is the character of the irreducible complex representation; $\rho: \mathbb{C}\tau \to \mathbb{C} \otimes_K M_n(D) \cong M_{mn}(\mathbb{C})$. Especially, the center of S_i is $K = \mathbb{Q}(\chi) = \chi(\mathbb{Q}\tau)$. Two characters χ_1 and χ_2 correspond to the same component S_i of $S(d)$ if and only if they are rationally equivalent, i.e. conjugated by the Galois group of K over $\mathbb{Q}$.

Thus we must classify up to rational equivalence the characters χ of τ which are injective on $\mathbb{Z}/d$. Each such χ is induced up from a linear character ψ of $\tau_2 = \mathbb{Z}/d \times \sigma_2$ faithful on $\mathbb{Z}/d$. Up to rational equivalence ψ is determined by $\operatorname{Ker}\psi \subset \sigma_2$; let X be the set of equivalence classes. The group $G = \sigma/\sigma_2$ acts on X by $g\psi(h) = \psi(ghg^{-1})$, $g \in G$, $h \in \tau_2$, and ψ_1 and ψ_2 induce isomorphic characters of τ if and only if $\psi_2 = g\psi_1$ for some $g \in G$. Thus the simple components of $S(d)$ correspond to

$$X/G \cong \{H \subset \sigma_2 \mid \sigma_2/H \text{ cyclic}\}/G.$$

Suppose $\psi \in X$ has kernel H and $\sigma_2/H = \mathbb{Z}/2^k$, then $\psi(\mathbb{Q}\tau_2) = \mathbb{Q}(\zeta_{2^k d})$. Let $\chi = \operatorname{Ind}_{\tau_2}^{\tau}(\psi)$. It has restriction $\chi|\tau_2 = \Sigma\{g\psi | g \in G\}$, and $g\psi$ is rationally equivalent to ψ if and only if $gHg^{-1} = H$. Denote by $G(\psi)$ the stabilizer of $\psi \in X$, $G(\psi) = \{g \in G \mid g(\operatorname{Ker}\psi)g^{-1} = \operatorname{Ker}\psi\}$. The representation ρ associated to χ splits up into $(G: G(\psi))$ inequivalent representations over τ_2, and $\rho(\mathbb{Q}\tau_2) = \Pi\mathbb{Q}(\zeta_{2^k d})$ with $(G : G(\psi))$ factors in the product. The simple component $\chi(\mathbb{Q}\tau)$ of $S(d)$ is the twisted group ring

$$S(\chi) = \rho(\mathbb{Q}\tau_2)^t G = (\Pi\mathbb{Q}(\zeta_{2^k d}))^t[G]$$

where G acts on $\mathbb{Q}\tau_2$ via the usual embedding $G \subset (\mathbb{Z}/d)^\times \times \operatorname{Aut}(\sigma_2)$. The center of $S(\chi)$ is

$$\mathbb{Q}(\chi) = \mathbb{Q}(\zeta_{2^k d})^{G(\psi)};\ G(\psi) \subset \mathbb{Z}/d \times \operatorname{Aut}(\sigma_2/H) = (\mathbb{Z}/2^k d)^\times .$$

The involution α on $\mathbb{Q}(\chi)$ is induced from $\alpha(\zeta_{2^k d}) = \zeta_{2^k d}^{-1}$ and is trivial precisely when $-1 \in G(\psi)$. Thus $(S(\chi),\alpha,1)$ has type O or type Sp if and only if

3.10 $$g_0 H^{-1} g_0^{-1} = H,\ H = \operatorname{Ker}\psi \text{ and } g_0 x g_0^{-1} = x^{-1} \text{ for } x \in \sigma_2/H.$$

where $g_0 \in \sigma$ maps to $-1 \in (\mathbb{Z}/d)^\times$. Note that the extension of σ_2/H by g_0 must then be either the dihedral group or the quaternion group of order 2^{k+1}. It follows that the image of g_0^2 under the embedding $\psi: \mathbb{Z}/2^k \to \mathbb{Q}(\zeta_{2^k})$ is ± 1.

To distinguish between type O and type Sp we count dimensions of the ± 1 eigenspaces $S(\chi)^\pm$ of α. If $\dim S(\chi)^+ > \dim S(\chi)^-$ we have type O and otherwise type Sp.

Let $g_1,\dots,g_s$ be elements in σ, representing $G(\psi) \subset G$. Consider the crossed product algebra of $\mathbb{Q}(\zeta_{2^k d})$ by $G(\psi)$

$$S'(\chi) = (\mathbb{Q}(\zeta_{2^k d})/\mathbb{Q}(\zeta_{2^k d})^{G(\psi)}, f)$$

where f is specified by $f(g_i,g_j) = \psi(h)g_k$ when $g_i g_j = h\, g_k$, $h \in \sigma_2$. Then $S(\chi)$ is the crossed product of $T = \Pi S'(\chi)$ over $\bar{G} = G/G(\psi)$. As $\bar{G}$ permutes freely the factors, $S(\chi)$ is actually a matrix ring over $S'(\chi)$. If $s \in T$, and $g \in \bar{G}$, then $\alpha(sg) = g^{-1}\alpha(s)g \cdot g^{-1}$. If $g^2 \neq 1$ in $\bar{G}$, then the subspaces $T\cdot g$ and $T\cdot g^{-1}$ are interchanged and contribute equally to the eigenspaces $S(\chi)^\pm$. Similarly, if $g \neq 1$ in $\bar{G}$ but $g^2 = 1$, then α preserves $T\cdot g$ but conjugation by g permutes freely the summands of $T\cdot g = \Pi S'(\chi)g$, so again the ± 1 eigenspaces have the same dimension

We are left with $T\cdot 1$, which contains $(\bar{G}:1)$ copies of $S'(\chi)$, all invariant under α. Thus $S(\chi)$ has the same type as $S'(\chi)$.

A similar check shows that in $S'(\chi)$ the subspaces $Q(\zeta_{2^k d})\cdot g$ contribute equally to $S^{\pm}(\chi)$ when $g \neq g_0$, since the conjugation $c(x) = g^{-1}\alpha(x)g$ is nontrivial on $Q(\zeta_{2^k d})$. For g_0 we get

$$\alpha(xg_0) = \alpha(x)^{g_0^{-1}}\chi(g_0^2)g_0 = x\psi(g_0^2)g_0.$$

Thus $Q(\zeta_{2^k d})g_0$ belongs totally to the $\chi(g_0^2)$-eigenspace of α, and $S(\chi)$ has type O if $\psi(g_0^2) = 1$ and type Sp when $\psi(g_0^2) = -1$.

The element $g_0^2 \in \sigma_2$ need not lie in the center C_2. But we conclude

Proposition 3.11. If σ_2 is abelian, then there is a Morita equivalence $(Q \otimes R_2(d),\alpha,1) \sim (Q \otimes C_2,\beta,T_0)$ where β is given in 3.9, and $T_0 \in C_2$ has the same image in $H^1(K_1(Q \otimes R_2(d)))$ as g_0^2

Corollary 3.12. In the situation of 3.9, suppose further that $H^1(C^\times) \to H^1((Q \otimes C)^\times)$ is injective. Then there is a Morita equivalence $(R_2(d),\alpha,1) \sim g_2\cdot(C,\beta,T_0)$.

We shall see in the next section that the assumption of 3.12 is satisfied for all cases of interest to us. But it is not true in general that $H^1(C^\times)$ injects into $H^1(C \otimes Q^\times)$. Indeed we have

Example 3.13. Consider the group $\tau = \mathbb{Z}/d \tilde{\times} D8$, where D8 is the dihedral group, and $t\colon D8 \to (\mathbb{Z}/d)^\times$ is given by $t(X) = -1$, $t(Y) = 1$ where $D8 = \langle X,Y \mid X^4 = 1,\ Y^2 = 1,\ YXY^{-1} = X^{-1}\rangle$. The ring $S(d)$ has one type U summand and two type O summands, so $H^1((C \otimes Q)^\times) = \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

On the other hand, suppose 2 is inertial in $Q(\zeta_d)/Q(\zeta_d+\zeta_d^{-1})$. The center C of $R_2(d)$ is $C = (\mathbb{Z}_2[\zeta_d][X^2,Y])^X$, and

$$H^1(C^\times) = H^1(\mathbb{Z}_2[\zeta_d+\zeta_d^{-1}][\mathbb{Z}/2]^\times) \oplus H^1(1+(1-X^2)C^\times).$$

The first factor is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ (corresponding to $H^1(C \otimes Q^\times)$). We show that the second factor is equal to $\mathbb{F} = \mathbb{Z}_2[\zeta_d+\zeta_d^{-1}]/(2)$. Indeed, $1+(1-X^2)C \cong 1+2A$ where $A = \{a_0+(a_1-\bar{a}_1)Y \mid a_0 \in \mathbb{Z}[\zeta_d+\zeta_d^{-1}],\ a_1 \in \mathbb{Z}_2[\zeta_d]\}$. There is an exact sequence

$$1 \to 1+4\mathbb{Z}_2[\zeta_d]^\times \to 1+2A \to \mathbb{F} \to 0$$

and since $1+4\mathbb{Z}_2[\zeta_d]^\times \cong \mathbb{Z}_2[\zeta_d]$ has trivial Galois cohomology, the claim follows.

We now give a partial calculation of the Rothenberg exact sequence for $R_2(d)$.

Proposition 3.14. The map

$$d_n\colon L_n^K(R_2(d),\alpha,1) \to H^n(K_1(R_2(d)))$$

is injective for $n = 0,1$. Suppose σ_2 is abelian. A sufficient condition that d_2 and d_3 be injective is that $(Q \otimes R_2(d),\alpha,1)$ contains a type Sp summand. For groups satisfying the assumptions of 3.12 this condition is also necessary.

Proof. We can assume that -1 lies in the image of $t\colon \sigma \to (\mathbb{Z}/d)^\times$. Otherwise both sides are zero. Consider the projection.

$$(R_2(d),\alpha,1) \to (\mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d][\sigma/\sigma_2],\alpha,1) \sim g_2\cdot(A,1,1)$$

where $A = \mathbb{Z}_2[\zeta_d]^{\sigma/\sigma_2}$. The injectivity for d_0, d_1 follows from [W3, Theorem 11]. If σ_2 is abelian we can use the Morita equivalence from 3.9, and consider

$$d_n\colon L_n^K(C,\beta,u) \to H^n(C^\times).$$

The existence of a type Sp summand is equivalent to a projection $(C\otimes Q,\beta,u) \to (E,1,-1)$. The non-trivial element in $L_2^K(C,\beta,u)$ is represented by the plane with $Q = \begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix}$ and d_2 maps it to the class of $1 + ((1-u)^2/u)b$ in $H^0(C^\times)$. Since $1 + 4b \neq 0$ in $H^0(E^\times)$ the result follows. Similarly, the non-trivial element in $L_3^K(C,\beta,u)$ is represented by $\begin{pmatrix} 0 & 1 \\ -u & 1 \end{pmatrix}$ with d_3-image $\mathrm{cls}(u)$ mapping to $-1 \in H^1(E^\times)$. Finally, $H^*(C\otimes Q^\times)$ is isomorphic to the product $H^*(E_i^\times)$ over the type O and type Sp factors in $(C\otimes Q,\beta,u)$. The last part follows. //

Remark 3.15. If $d_2 = 0$ above we have a surjection $L_2^X(R_2(d)) \twoheadrightarrow g_2\cdot L_2^K(A)$ and get g_2 elements $\langle c\rangle$ in $L_2^X(R_2(d))$. Each have order 2 (cf. [W3, p.67]) and they come from $L_2^X(\mathbb{Z}\pi)(d)$ in 2.21. If $d_3 = 0$ the surjection $L_3^X(R_2(d)) \to g_2 L_3^X(A) \to g_2 L_3^K(A)$ gives g_2 elements $\langle\tau\rangle$ in $L_3^X(R_2(d))$. Each generate a $\mathbb{Z}/4$ since $2\langle\tau\rangle = \langle\tau^2\rangle = t_3(-1)$ and $-1 \in H^0(A^\times)$ is non-trivial because $A/\mathbb{Z}_2$ is unramified. Thus we get g_2 copies of $\mathbb{Z}/4$ in $L_3^X(\mathbb{Z}\pi)(d)$.

We return to the calculation of the exact sequence 2.23 (σ_2 abelian). We have $L_n^X(R_2(d)) = L_n^X(C_2,\beta,u)$ by 3.9, and

$$C_2 \otimes Q = \Pi E_{i,2}$$

where ΠE_i is the center of $Q[\zeta_d]^t\sigma$. Let A_i be the integers in E_i with completions $A_{i,2} = \prod_{y|2} A_{i,y}$. The natural inclusion

of C_2 in $C_2 \otimes Q$ embeds into $\Pi A_{i,2}$ and induces a map

3.16 $$\varphi\colon H^{n+1}(C_2^\times) \to \Pi H^{n+1}(A_{i,2}^\times)$$

which is related to the evaluation of ψ_n in 2.23 by the commutative diagram

3.17 $$\begin{array}{ccccc} & \overset{\cong}{\nearrow} & H^{n+1}(C_2^\times) \xrightarrow{\varphi} \Pi H^{n+1}(A_{i,2}^\times) & \rightarrowtail & \Pi H^{n+1}(E_{i,2}^\times) \\ H^{n+1}(K_1(R_2(d))) & & \downarrow \bar{t}_n \qquad\qquad \vdots & & \downarrow \Pi p_{i,n} \\ & \underset{t_n}{\searrow} & L_n^X(R_2(d)) \xrightarrow{\bar{\psi}_n} CL_n(S(d)) & \xrightarrow{\cong} & \Pi CL_n(S_i) \end{array}$$

We have suppressed the antistructures, $R_2(d) = (R_2(d),\alpha,1)$, $C_2 = (C_2,\beta,u)$ etc. The simple components S_i have center E_i. The value of $CL_n(S_i)$ is given in 2.19 and $p_{i,n}$ is given in 2.20: if S_i (or rather $(S_i,\alpha,1)$) has type O then $p_{i,1}$ and $p_{i,2}$ are the natural homomorphisms into $H^0(C(E_i))$ and $H^1(C(E_i))$, respectively. Finally, if S_i has type Sp then we get the same results but with n replaced by $n+2$.

Note from 2.15 and 3.14 that if $(S(d),\alpha,1)$ contains a type Sp summand then $\bar{t}_n$ in 3.17 is surjective with kernel $g_2\cdot(\mathbb{Z}/2)$.

Detailed calculations for groups with periodic cohomology are presented in the next two paragraphs.

§4. The map $\bar{\psi}_n: L_n^X(R_2(d)) \to CL_n(S(d))$.

Let τ be a 2-hyperelementary group with periodic cohomology and let $S(d) = Q(\zeta_d)^t[\sigma]$ be a factor in $Q\tau$ for which $t: \sigma \to (\mathbb{Z}/d)^\times$ has abelian kernel. The corresponding factor in $\mathbb{Z}_2\tau$ is $R_2(d) = \mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]^t\sigma$.

Suppose throughout the paragraph that $-1 \in \text{Image}(t)$; otherwise $L_i^X(R_2(d)) = 0$. Choose $g_0 \in \sigma$ with $t(g_0) = -1$. Let $\sigma_1 = t^{-1}\langle 2\rangle$ and $\sigma_2 = \text{Ker } t$. Note that $g_0 \in \sigma_1$ if and only if $-1 \in \langle 2\rangle$.

The ring $\mathbb{Z}_2 \otimes \mathbb{Z}[\zeta_d]$ decomposes into a product of isomorphic rings. Denote by B one component, $B = \mathbb{Z}_2[\zeta_d]$, and let $A = B^{\sigma_1}$. The center C_2 of $R_2(d)$ decomposes into $g_2 = ((\mathbb{Z}/d)^\times : \langle 2, t(\sigma)\rangle)$ components, $C_2 = g_2 \cdot C$ and $C = (B\sigma_2)^{\sigma_1}$ where σ_1 acts on B as above and on σ_2 by conjugation. We show below in 4.14 that $H^1(C^\times)$ embeds into $H^1(Q \otimes C^\times)$. By 3.12 there is a Morita equivalence

4.1 $$(R_2(d), \alpha, 1) \sim g_2 \cdot (C, \beta, T_0)$$

where $\beta = g_0^{-1}\alpha g_0 | C$ and $T_0 = g_0^2$ when $g_0^2 \in C$. We now give a more precise description of the center.

If $\sigma = \mathbb{Z}/2^k$ then σ_1 acts trivially on σ_2, so $C = A\sigma_2$. If $\sigma = Q2^k$ (presented as in 1.1) then $\sigma_1/\sigma_2 \subseteq \mathbb{Z}/2 \oplus \mathbb{Z}/2$. Let $B_0 = B^{\sigma_1 \cap \langle X\rangle}$. Then $C = (B_0\sigma_2)^G$ where $G = \sigma_1/\sigma_2 \cdot (\sigma_1 \cap \langle X\rangle) \subseteq \mathbb{Z}/2$. and $A = B_0^G$. If $G = \{1\}$ then $C = A\sigma_2$ as above. Otherwise B_0/A is a quadratic unramified extension of 2-adic rings. We choose $\theta \in B_0$ such that

4.2 $$\theta + \bar{\theta} \in A^\times,\ \theta\bar{\theta} = 1,\ B_0 = A[\theta]$$

where $\bar{\theta} = \theta^Y$. (This is possible by Hensels lemma because $x^2 + ax + 1$ is irreducible over $\mathbb{F}_{2^n}$ for suitable $a \in \mathbb{F}_{2^n}^\times$). Let

$T \in \sigma_2$ be a generator ($T = X$ or X^2). Then $C \subset B_0\sigma_2$ is the free A-module with generators

4.3 $$1,\ T^{2^{r-2}},\ T^i + T^{-i},\ \theta T^i + \bar{\theta}T^{-i},\ 1 \le i \le 2^{r-2} - 1.$$

where $(\sigma_2 : 1) = 2^{r-1}$ and $T^{2^{r-1}} = 1$. This ring is denoted $C_A(r)$. Spelling out 3.12 in our situation where $\sigma = \mathbb{Z}/2^k$ or $\sigma = Q2^k$ we have:

<u>Proposition 4.4</u>. (i) Let $\sigma = Q2^k$, $\sigma_1 \subseteq \langle X\rangle$ and $g_0 \notin \sigma_1$. Then $(R_2(d), \alpha, 1) \sim g_2 \cdot (A\sigma_2, 1, g_0^2)$.

(ii) Let either $\sigma = \mathbb{Z}/2^k$ or $\sigma = Q2^k$ with $\sigma_1 \subseteq \langle X\rangle$ and $g_0 \in \sigma_1$. Then $(R_2(d), \alpha, 1) \sim g_2 \cdot (A\sigma_2, \beta, g_0^2)$ with $\beta|A = 1$ and $\beta(h) = h^{-1}$ for $h \in \sigma_2$.

(iii) Let $\sigma = Q2^k$, $\sigma_1 \nsubseteq \langle X\rangle$ and $g_0 \notin \langle X\rangle$. Suppose further that $(\sigma_2 : 1) = 2^{r-1} \ge 4$. Then $(R_2(d), \alpha, 1) \sim g_2 \cdot (C_A(r), 1, g_0^2)$.

(iv) Let $\sigma = Q2^k$, $\sigma_1 \nsubseteq \langle X\rangle$ and $g_0 \in \langle X\rangle$. Suppose further that $(\sigma_2 : 1) = 2^{r-1} \ge 4$. Then $(R_2(d), \alpha, 1) \sim g_2 \cdot (C_A(r), \beta, T_0)$ where $\beta|A = 1$, $\beta(\theta) = \theta$ and $\beta(T^i) = T^{-i}$.

We note if $\sigma_2 \subseteq \langle X^2\rangle$ that σ_1 is a proper subgroup of σ in the quaternion cases, since σ_1/σ_2 must by cyclic. Thus in 4.4(iv), $-1 \notin \langle 2\rangle$. Note also that $g_0 \equiv X \pmod{\langle X^2\rangle}$ so $g_0^2 \notin C_A(r)$ in this case.

Write $\tau = \mathbb{Z}/d \tilde{\times} \sigma$ (with twisting t). The groups τ which can occur under our assumptions (σ_2 cyclic, $-1 \in \text{Image } t$) are as follows

4.5

$$\tau = \mathbb{Z}/d \tilde{\times} \mathbb{Z}/2^k \quad (\sigma_2 = \mathbb{Z}/2^{k-1};\ g_0^2 \text{ the generator})$$

$$\tau = Q(2^k a) \quad (\sigma_2 = \langle X \rangle;\ g_0^2 = X^{2^{k-2}})$$

$$\tau = Q(2^k a;b,1) \quad (\sigma_2 = \langle X^2 \rangle;\ g_0^2 = X^{2^{k-2}})$$

$$\tau = Q(2^k;b,c) \quad (\sigma_2 = \langle X^2 \rangle;\ g_0^2 = X^2)$$

(4.4(iv) corresponds to $\tau = Q(2^k;b,c)$, $b \geq c > 1$, $-1 \notin \langle 2 \rangle \subset (\mathbb{Z}/bc)^{\times}$).

In all cases, except for $\mathbb{Z}/d \tilde{\times} \mathbb{Z}/2^k$ with $\sigma_2 = 1$, $S(d)$ contains type Sp summands, and 3.14, 3.15 give

Proposition 4.6.

(i) If $\sigma_2 \neq 1$, $L_i^X(R_2(d)) \cong g_2 \cdot (H^{i+1}(C^{\times})/(\mathbb{Z}/2))$ where C (with involution) is listed in 4.4.

(ii) If $\sigma_2 = 1$, $L_i^X(R_2(d)) = g_2 \cdot L_i$, and

$$L_i = 0;\ H^0(A^{\times});\ \mathbb{Z}/2 \oplus H^1(A^{\times});\ H^0(A^{\times})/\langle 1-4b,-1 \rangle \oplus \mathbb{Z}/4$$

for $i = 0,1,2,3$.

We now calculate $H^{i+1}(C^{\times})$ for the antistructures in 4.4. Details are given in the cases (iii) and (iv), the easier cases (i) and (ii) are left for the reader.

Consider the central ideal $(1-T^{2^{i-1}})$ in $C_A(i+1)$. The quotient $C_A(i+1)/(1-T^{2^{i-1}})$ is $C_A(i)$, since the basis from 4.3 maps onto the basis for $C_A(i)$ except for the element $T^{2^{i-2}} \in C_A(i)$ which is the image of

4.7 $$T_{i-2} = \frac{\theta T^{2^{i-2}} + \bar{\theta} T^{-2^{i-2}}}{\theta + \bar{\theta}} \in C_A(i+1).$$

Using notation as in 4.2, let

4.8 $$A_i = (B_0[\zeta_{2^i}])^{\mathbb{Z}/2},$$

where $\mathbb{Z}/2 \subset G \times (\mathbb{Z}/2^i)^{\times}$ is diagonally embedded. Its quotient field is a dyadic completion of $Q(\zeta_{2^i d} + \zeta_{2^i d}^{-1})$. Setting $T = \zeta_{2^i}$ defines -

4.9 $$\varphi_i : C_A(i+1) \to A_i$$

and φ_i induces an isomorphism between the ideal $(1-T^{2^{i-1}})$ and $2A_i$. Note in case 4.4(iv), that the involution β on $C_A(i+1)$ corresponds to the Galois automorphism of A_i induced from $-1 \in (\mathbb{Z}/2^i)^{\times}$. Thus $\beta|A_0 = 1$ and $\beta|A_1 = 1$ in all cases.

We have a string of epimorphisms

$$C_A(r)^{\times} \to C_A(r-1)^{\times} \to \dots \to C_A(2)^{\times} \to C_A(1)^{\times}$$

and $\mathrm{Ker}(C_A(i+1)^{\times} \to C_A(i)^{\times}) \cong 1 + 2A_i^{\times}$, cf. 2.5. Define a filtration of $C_A(r)^{\times}$ by

$$F^i = \mathrm{Ker}(C_A(r)^{\times} \to C_A(i)^{\times}),\quad F^i/F^{i+1} = 1 + 2A_i^{\times}.$$

There is a corresponding spectral sequence $\{E_k^{i,j}, d_k\}$ converging to $H^*(C_A(r)^{\times})$. It is periodic in j of period 2, $\{E_k^{*,j}, d_r\} \cong \{E_k^{*,j+2}, d_r\}$, $j \in \mathbb{Z}$ and the E_1-term is

$$E_1^{i,j} = \begin{cases} H^{i+j}(1+2A_i^{\times}); & 0 \leq i \leq r-1 \\ 0 & ;\ i \geq r \end{cases}$$

Lemma 4.10. Let $\mathbb{F}$ be the common residue field of the A_i, $\mathbb{F} = A_0/2A_0$.

(a) If $\beta|A_i = 1$ then $H^0(1+2A_i^{\times}) = \mathbb{F} \oplus \mathbb{Z}/2$, $H^1(1+2A_i^{\times}) = \mathbb{Z}/2$.

(b) If $\beta|A_i \neq 1$ then $H^*(1+2A_i^\times) = \mathbb{F}\ (i \geq 2)$

<u>Proof</u>. In both cases, $H^*(1+2A_i^\times)$ is calculated from the exact cohomology sequence of

$$0 \to 1+2\pi_i A_i^\times \to 1+2A_i^\times \to \mathbb{F} \to 0$$

where π_i is the prime element of A_i. The logarithm defines an (involutive) isomorphism

$$1+2\pi_i A_i^\times \cong 2\pi_i A_i \cong \pi_i A_i .$$

In case a), $H^*(\pi_i A_i) \cong H^*(A_i)$ and $\delta: H^1(\mathbb{F}) \to H^0(1+2\pi_i A_i^\times)$ can be identified with $p: \mathbb{F} \to \mathbb{F}$, $p(x) = x + x^2$.
Case b) is more involved. We must show that $1+2\pi_i A_i^\times$ is cohomological trivial (w.r.t. β).

Let, $E_i = ff(A_i)$, $i \geq 2$. Using transitivity of differents and the well-known different of $Q_2(\zeta_{2^i})/Q_2(\zeta_{2^i}+\zeta_{2^i}^{-1})$ it follows that E_i/E_i^β has different (π_i^2). Thus $\pi_i + \beta(\pi_i) = p_i$ is a prime element of E_i^β. Now, A_i is a free A_i^β module with generators 1 and $\beta(\pi_i)$ and $\pi_i A_i = p_i A_i^\beta + \pi_i A_i^\beta$. A simple calculation gives $H^0(\pi_i A_i) = 0$, and hence $H^0(1+2\pi_i A_i^\times) = 0$. On the other hand

$$|H^0(1+2\pi_i A_i^\times)|/|H^1(1+2\pi_i A_i^\times)| = |H^0(A_i^\times)|/|H^1(A_i^\times)|$$

by the invariance of Herbrand quotient under finite extensions, and $|H^0(A_i^\times)| = |H^1(A_i^\times)|$. Indeed, the homology sequence of $0 \to A_i^\times \to E_i^\times \to \mathbb{Z} \to 0$ takes the form

$$0 \to H^0(A_i^\times) \to H^0(E_i^\times) \to \mathbb{Z}/2\mathbb{Z} \to H^1(A_i^\times) \to 0$$

and $H^0(E_i^\times) = \mathbb{Z}/2$ by class field theory. Thus $H^*(1+2\pi_i A_i^\times) = 0$ and b) follows. //

Before we calculate d_1 in $E_1^{*,*}$, a few words about A_i are in order. For $k \geq 1$ we choose prime element

$$\pi_i = 1 - \frac{\zeta_{2^i}\theta + \zeta_{2^i}^{-1}\bar{\theta}}{\theta + \bar{\theta}}$$

in A_i. We have $\pi_1 = 2$ and π_{i+1} satisfies the Eisenstein equation

4.11 $$\pi_{i+1} - 2\pi_{i+1} + (\pi_i - a_i) = 0, \quad a_i = (1-\zeta_{2^i}-\zeta_{2^i}^{-1})/(\theta+\bar{\theta})^2 .$$

Note that a_i has valuation $v(a_i) = 2$ since $\theta+\bar{\theta}$ is a unit and $2-\zeta_{2^i}-\zeta_{2^i}^{-1}$ is a prime element of $\mathbb{Z}_2[\zeta_{2^i}+\zeta_{2^i}^{-1}]$ which is a subring of A_i of ramification index 2. We have $\pi_{i+1}^2 \equiv \pi_i \pmod{\pi_i^2}$ for $i \geq 2$ and $\pi_2^2 \equiv 2(1+\pi_2) \pmod 4$. Hence

4.12 $$\pi_i^{e/2} \equiv \pi_2, \quad \pi_i^e/2 \equiv \pi_2^2/2 \equiv 1+\pi_2 \pmod 2$$

where $i \geq 2$ and $e = e(A_i) = 2^{i-1}$.

<u>Lemma 4.13</u>. (a) In case 4.4(iii), $\beta|A_i = 1$ for $i \geq 0$. The differential $d_1: E_1^{i,j} \to E_1^{i+1,j}$ is injective if $i+j \equiv 1 \pmod 2$ and $1 \leq i \leq r-1$, and $(d_1\ E_1^{0,*}) = 0$.

(b) In case 4.4(iv), $\beta|A_i \neq 1$ for $i \geq 2$ but $\beta|A_0 = \beta|A_1 = 1$. The differential $d_1: E_1^{i,j} \to E_1^{i+1,j}$ maps isomorphically for $i \geq 2$, $d_1|E_1^{1,2j-1}$ is surjective and $d_1|E_1^{0,*} = 0$.

<u>Proof</u>. In case a), the generator in $E_1^{i,j}$ is -1 which corresponds to $1-(1-X^{2^{i-1}}) = X^{2^{i-1}} \in C_A(i+1)^\times$ and lifts to $X_{i-1} \in C_A(i+2)^\times$. Thus

$$d_1(-1) = \varphi_{i+1}(X_{i-1}^2) = (1-\pi_2)^2 = 1+2\gamma$$

where $\gamma = -1 + 2(\theta+\bar{\theta})^{-2}$. The class of $1+2\gamma$ is non-trivial in $H^0(1+2A_{i+1}^{\times})$.

We turn to case b). Let $a \in A_0$ with reduction $\tilde{a}$ to $\mathbb{F}$. Consider $[1+2a]_i \in H^0(1+2A_i^{\times})$. It has differential

$$\begin{aligned} d_1([1+2a]_i) &= \varphi_{i+1}([1+(1-X_{i-1})a]^{-1}[1+(1-\beta(X_{i-1}))a]) \\ &= (1+\pi_2 a)^{-1}(1+\beta(\pi_2)a) \\ &= 1+2(1-\pi_2)a \pmod{2\pi_2} \\ &= [1+2a]_{i+1} \end{aligned}$$

Thus $d_1(\bar{a}) = \bar{a}$. The rest is clear. //

The spectral sequence has $E_2 = E_\infty$ in both case 4.4(iii) and 4.4(iv). This is formal. Indeed, if $i+j = 1$ the only non-trivial $E_2^{i,j}$ are $E_2^{0,1} = \mathbb{Z}/2$ and $E_2^{r-1,2-r} = \mathbb{Z}/2$ in case (iii) and $E_2^{0,1} = \mathbb{Z}/2$ and $E_2^{1,0} = \mathbb{Z}/2$ in case (iv).

But $H^1(C_A(r)^{\times})$ must have rank ≥ 2, since -1 is always a non-trivial element and there must be at least one further element since $(C_A(r),\beta,u)$ contains both a type O and a type Sp factor. (In case (iii) the extra element is $g_0^2 = T^{2^{r-2}}$, in case (iv) it is T_0 which is not so easy to list explicitly).

The calculations in cases 4.4(i) and (ii) are quite analogous but easier. Instead of A_i one consider the ring $\tilde{A}_i = A[\zeta_{2^i}]$. $(\tilde{A}_1 = \tilde{A}_0 = A)$.

<u>Theorem 4.14</u>. In the situation of 4.4 one has

$$H^1(C^{\times}) = \langle -1 \rangle \oplus \langle T_0 \rangle,$$

where $T_0 = g_0^2$ except in case (iv). The groups $H^0(C^{\times})$ are (with corresponding numbering)

(i) $\prod_{i=2}^{r-1} H^0(1+2\tilde{A}_i^{\times})/\langle -1 \rangle \times H^0(1+2\tilde{A}_1^{\times}) \times H^0(\tilde{A}_0^{\times})$

(ii) $H^0(1+2\tilde{A}_{r-1}^{\times}) \times \mathbb{Z}/2 \times H^0(\tilde{A}_0^{\times})$, $r \geq 3$

(iii) $\prod_{i=2}^{r-1} H^0(1+2A_i^{\times})/\langle 1+2\gamma \rangle \times H^0(1+2A_1^{\times}) \times H^0(A_0^{\times})$, $r \geq 3$

(iv) $H^0(1+2A_{r-1}^{\times}) \times \mathbb{Z}/2 \times H^0(A_0^{\times})$, $r \geq 3$

In (ii), (iv) the $\mathbb{Z}/2$ is generated by $1+4b$ in $H^1(1+2A_1^{\times})$, where $\bar{b} \in \mathbb{F}$ not of the form $x+x^2$.

We proceed to evaluate $\bar{\Psi}_n\colon L_n^X(R_2(d)) \to CL_n(S(d))$. First, we dispose of the exceptional case 4.6(ii) where $S(d)$ contains no type Sp summands. In this case $S(d)$ is simple (with center $E = Q(\zeta_d)^{\mathbb{Z}/2^k}$, and (cf. 2.19)

$$CL_1(S(d)) = H^0(C(E)),\ CL_2(S(d)) = H^1(C(E)).$$

We have $A_2^{\times} \subset E_2^{\times} \subset C(E)$ and $H^*(A_2^{\times}) \subset H^*(C(E))$, where $A_2^{\times} = g_2 \cdot A^{\times}$.

<u>Proposition 4.15</u>. In the situation of 4.6(ii), $\bar{\Psi}_n$ is non-trivial only for $n = 1,2$. More precisely,

$$\bar{\Psi}_1\colon g_2 \cdot H^0(A^{\times}) \to H^0(C(E))$$

$$\bar{\Psi}_2\colon g_2 \cdot H^1(A^{\times}) \to H^1(C(E))$$

are the natural inclusions, and $\bar{\Psi}_2$ has kernel $g_2 \cdot \mathbb{Z}/2 \langle c \rangle$.

Proof. The only part which is not completely obvious is that the g_2 Arf invariant elememnts in L_2^X lies in the kernel of $\bar{\psi}_2$. But each element is represented by $Q = \begin{pmatrix} 1 & 1 \\ 0 & b \end{pmatrix}$. Its image in $L_2(E_y)$, $y|2$ is the plane with bilinear form $Q - Q^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Over fields (of char $\neq 2$) the bilinear form determines the quadratic form, so the image of c in $L_2(E_y)$ is represented by the hyperbolic form, hence $\bar{\psi}_2(c) = 0$. //

Suppose $S(d)$ contains both type O and type Sp summands. Then $L_n^X(R_2(d)) \cong g_2 \cdot (H^{n+1}(C^\times)/(\mathbb{Z}/2))$, cf. 3.17 and the rest of §3. The map $\bar{\psi}_n = \psi_n | L_n^X(R_2(d))$ is determined from the diagram

4.16
$$\begin{array}{ccc} g_2 \cdot L_n^X(C,\beta,T_0) & \xrightarrow{\bar{\psi}_n^{Sp}} & \prod_{\text{type Sp}} CL_n(E_i) \\ \uparrow & & \uparrow p_n^{Sp} \\ g_2 \cdot (H^{n+1}(C^\times)/\mathbb{Z}/2) & \xrightarrow{\varphi^{Sp}} & g_2 \cdot (\prod_{\text{type Sp}} H^{n+1}(A_{i,y}^\times)) \end{array}$$

and a corresponding diagram involving $\bar{\psi}_n^O$, φ^O. Here $\varphi_O = \prod_{\text{type O}} \varphi_i$, $\varphi_{Sp} = \prod_{\text{type Sp}} \varphi_i$. We note from 2.16-2.20:

$$\operatorname{Ker} p_n^{Sp} = 0 \text{ and } p_n^O = 0 \text{ for } n = 0,3$$

$$p_n^{Sp} = 0 \text{ and } \operatorname{Ker} p_n^O = 0 \text{ for } n = 1,2.$$

The maps φ^{Sp}, φ^O are induced from the projections

$$\varphi_i\colon C_A(r)^\times \to C_A(i+1)^\times \to A_i^\times$$

$$\varphi_i\colon A[\mathbb{Z}/2^{r-1}]^\times \to A[\mathbb{Z}/2^i]^\times \to A_i^\times$$

which map T to ζ_{2^i} (cf. 4.9).

Proposition 4.17. In the situation of 4.4 with $\sigma_2 \neq 1$, $\bar{\psi}_0^{Sp}$ and $\bar{\psi}_2^O$ are injective and $\bar{\psi}_3^{Sp}$ is surjective. Moreover, the rank (over $\mathbb{Z}/2$) of $\operatorname{Ker} \bar{\psi}_3^{Sp}$ is

$$\operatorname{rk}_2(\operatorname{Ker} \bar{\psi}_3^{Sp}) = \begin{cases} g_2 \cdot \prod_{i=0}^{r-2} |E_i : Q_2| & \text{for case 4.4(i),(iii)} \quad (r \geq 2) \\ g_2 \cdot |E_0 : Q_2| & \text{for case 4.4(ii),(iv).} \end{cases}$$

Proof. In each case of 4.14, there is precisely one type Sp summand. The domain for $\bar{\psi}_0^{Sp}$ is $H^1(C^\times)/(\mathbb{Z}/2) = \mathbb{Z}/2$ with generator T_0, which maps to -1 in the type Sp factor. This proves the first part of a) and the second part is similar.

We prove b). In case 4.4(iii), consider the ideal $J = (1+T^{2^{r-2}}) \subset C_A(r)$. We have the exact sequence from 2.5,

$$1 \to 1+(1+T^{2^{r-2}})^\times \to C_A(r)^\times \to A_{r-1}^\times \to 1$$

and hence $\varphi_{r-1}\colon H^0(C_A(r)^\times) \to H^0(A_{r-1}^\times)$ is surjective. Since A_{r-1} is the type Sp summand in 4.4(iii), this proves b). In case 4.4(iv), A_1 is the type Sp summand and we consider the projection

$$H^0(C_A(r)^\times) \xrightarrow{\varphi_1} H^0(A_1^\times).$$

In $H^0(C_A(r)^\times)$ we have the elements $1+(T+T^{-1})a$, $a \in A_0$, and $\varphi_1(1+(T+T^{-1})a) = 1-2a$. Since A_1 is unramified, $H^0(A_1^\times) \cong H^0(1+2A_1^\times)$ and φ_1 is surjective. The cases 4.4(i) and 4.4(ii) are treated similarly.

Finally, the rank of $\operatorname{Ker} \bar{\psi}_3^{Sp}$ follows from 4.14 because for a 2-ring A_i,

$$\operatorname{rk}(A_i^\times/A_0^{\times 2}) = |E_i : Q_2| + 1 = \operatorname{rk}((1+2A_i)^\times/(1+2A_i^\times)^2)$$

where $E_i = ff(A_i)$. Indeed, the torsion in $A_i^\times$ is cyclic (of even order) and the free part of $A_i^\times$ is isomorphic to the free part of $(1+2\pi_i A_i)^\times \cong A_i^+$. This has rank $|E_i:Q_2|$. //

We end this paragraph with a straight-forward but lengthy analysis of the homomorphisms

4.18
$$\varphi\colon H^0(C_A(k)^\times) \to H^0(A(k)^\times)$$
$$\varphi\colon H^0(A[\mathbb{Z}/2^{k-1}]^\times) \to H^0(A(k)^\times)$$

in the cases of trivial involution (4.4(i), 4.4(iii)). Here $A(k)$ equals $\prod_{i=0}^{k-1} A_i$ or $\prod_{i=0}^{k-1} \tilde{A}_i$, and $\varphi = \Pi\,\varphi_i$. The results imply a calculation of $\bar{\varphi}_1^0$, needed in the next paragraph.

First we need

<u>Lemma 4.19</u>. Let A be a 2-adic ring with even ramification index e and trivial involution. Let π be a prime element and let $\mathbb{F}$ be the residue field of A. Choose a representative system F for $\mathbb{F}$ in A. Then $f\colon H^0(1+2A^\times) \to H^0(A^\times)$ has

$$\text{Ker } f \cong \{(1+c_{e/2}\pi^{e/2}+c_{e/2+1}\pi^{e/2+1}+\ldots+c_{e-1}\pi^{e-1})^2 \mid c_i \in F\} \cong \mathbb{F}^{e/2}$$

$$\text{Coker } f \cong \{1+c_1\pi+c_3\pi^3+\ldots+c_{e-1}\pi^{e-1} \mid c_i \in F\} \cong \mathbb{F}^{e/2}.$$

<u>Proof</u>. Since $A^\times/1+\pi A \cong \mathbb{F}^\times$ has odd order $H^0(A^\times) \cong H^0(1+\pi A^\times)$. The elements of $1+\pi A^\times$ can be expanded into series $1+\sum_{i=1}^{\infty} c_i\pi^i$ and the quotient $1+\pi A^\times/1+2A^\times$ is represented by the partial sums $1+c_1\pi+\ldots+c_{e-1}\pi^{e-1}$. This follows because $1+a\pi+2b = ((1+a\pi)(1+2b/(1+a\pi)) \equiv 1+a\pi \pmod{1+2A}$.

We have an exact sequence

$$0 \to H^1(1+\pi A^\times/1+2A^\times) \to H^0(1+2A^\times) \xrightarrow{f} H^0(1+\pi A^\times) \to H^0(1+\pi A^\times/1+2A^\times) \to 0$$

Hence Ker f and Coker f are the cohomology groups of $1+\pi A^\times/1+2A^\times$. Now,

$$(1+c_1\pi+\ldots+c_{e-1}\pi^{e-1})^2 = 1+c_1^2\pi^2+c_2^2\pi^4+\ldots+c_{e/2-1}^2\pi^{e-2} \in 1+\pi A^\times/1+2A^\times$$

Squaring is a bijection of $\mathbb{F}$, hence also of $F \pmod 2$. Any series with only even coefficients is thus a square in $1+\pi A/1+2A$, and $(1+\Sigma c_i\pi^i)^2 = 1$ if and only if $c_1 = c_2 = \ldots = c_{e/2-1} = 0$. //

We filter the ring $A(k)^\times = \prod_{i=0}^{k-1} A_i^\times$ compatibly with the filtration of $C_A(k)$ above, that is, by the kernels $F^n = \text{Ker}\{H^0(A(k)^\times) \to H^0(A(n)^\times)\}$. The map φ preserves filtrations. Consider the filtered complex

$$1 \to C_A(k)^\times \xrightarrow{Sq} C_A(k)^\times \to H^0(A(k)^\times) \to 1.$$

Its homology is equal to $\hat{E}_\infty = H^1(C_A(k)^\times) \oplus \ker\varphi \oplus \text{Coker}\,\varphi$, and there is a spectral sequence $(\hat{E}_r, d_r)$ with

4.20
$$\hat{E}_1^n = \langle -1\rangle \oplus \text{Ker } f_n \oplus \text{Coker } f_n$$

converging to $\hat{E}_\infty$, where $f_n\colon H^0(1+2A_n^\times) \to H^0(A_n^\times)$. Indeed, $\hat{E}_0^n = (1+2A_n^\times) \oplus (1+2A_n^\times) \oplus H^0(A_n^\times)$ and $d_0(c_0,c_1,a) = (0,c_0^2,c_1)$, giving the stated E_1-term. The differential $d_1\colon \hat{E}_1^{n-1} \to \hat{E}_1^n$ was computed on -1 in the course of proving 4.13. It vanishes for $n=1$, k and for $2 \le n \le k-1$,

$$d_1(-1) = (1-\pi_2)^2 = (1+\pi_n^{e/2})^2 \in \text{Ker } f_n.$$

For trivial reasons $d_1\colon \text{Ker } f_n \to \text{Coker } f_{n+1}$ vanishes when $n = 0,1,k-1$. Otherwise it is a mapping $d_1\colon \mathbb{F}^{e/2} \to \mathbb{F}^e$ where $e = e(A_n/A_0) = 2^{n-1}$.

<u>Lemma 4.21</u>. Let $e = 2^{n-1}$ be the ramification index of A_n. Then d_1: Ker $f_n \to$ Coker f_{n+1} is $\mathbb{F}$-linear of rank $e/4$ unless $n = 2$ when $d_1: \mathbb{F} \to \mathbb{F}^2$ is $d_1(c) = (0, c^2+c)$. More precisely when $n \geq 3$, the kernel of d_1 in $H^0(1+2A_n^{\times})$ consists of the elements $(1+\Sigma c_i\pi_n^i)^2$ with $e/2 \leq i \leq e$, i even; the cokernel is the quotient of $H^0(A_{n+1}^{\times})$ represented by $1+\Sigma c_i\pi_{n+1}^i$ with $1 \leq i \leq (3/2)e$, i odd.

<u>Proof</u>. It is easily seen that d_1: Ker $f_n \to$ Coker f_{n+1} is given as

$$d_1([1+2\alpha_n]) = [1+\pi_2\alpha_{n+1}] \in 1+2A_{n+1}^{\times}$$

where $\pi_n \in A_n$ and α_{n+1} is obtained from α_n by replacing ζ_{2^n} with $\zeta_{2^{n+1}}$. By 4.19, Ker f_n is represented by classes $(1+\Sigma c_i\pi_n^i)^2$, $e/2 \leq i \leq e$, with $c_i \in A_0$. It suffices to calculate $d_1(1+c\pi_n^i)^2$ and using 4.12 we get

$$d_1(1+c\pi_n^i)^2 = 1+\pi_2(c\pi_{n+1}^i+c^2\pi_{n+1}^{2i-e}(1+\pi_{n+1}^{e/2}))$$
$$= 1+\pi_{n+1}^e(c\pi_{n+1}^i+c^2\pi_{n+1}^{2i-e}(1+\pi_{n+1}^{e/2})).$$

The target Coker f_{n+1} is the quotient of $H^0(A_{n+1}^{\times})$ represented by $1+\Sigma c_i\pi_{n+1}^i$ where $1 \leq i < 2e$ is odd, and even more, we can compute modulo even powers of π_{n+1} (cf. the proof of 4.19). If $n = 2$ then $e = 2$ and

$$d_1((1+c\pi_2)^2) = 1+(c^2+c)\pi_3^3,$$

so that $d_1: \mathbb{F} \to \mathbb{F}^2$ has kernel $\mathbb{F}_2$ and cokernel $\mathbb{F} \oplus \mathbb{F}/p(\mathbb{F})$, $p(x) = x^2+x$. If $n > 2$ then $2i-e$ and $e/2$ are even and all what remains is

$$d_1((1+c\pi_n^i)^2) = 1+c\pi_{n+1}^{e+i}.$$

Thus the kernel of d_1 is Ker f_n^{ev} consisting of $(1+\Sigma c_i\pi_n^i)^2$ with $e/2 \leq i \leq e$, i even, and d_1 maps Ker f_n^{od} injectively with image $1+\Sigma c_j\pi_{n+1}^j$, $3/2e < j < 2e$, j odd. //

<u>Lemma 4.22</u>. In the spectral sequence $\hat{E}_r$ above, $d_p = 0$ when $p \geq 2$.

<u>Proof</u>. Assume inductively that $E_n^p = E_n^2$. We must evaluate d^p on the elements of $H^0(1+2A_n^{\times})$ represented by $(1+c\pi_n^i)^2$, $e/2 \leq i \leq e$, i even. As in the proof of 4.21,

$$d_p((1+c\pi_n^i)^2) = 1+\pi_{p+1}(c\pi_{p+n}^i+c^2\pi_{p+n}^{2i-e}(1+\pi_{p+n}^{e/2})).$$

Since $n \geq 3$ all exponents of π_{p+n} are even, so to complete the proof we must show that the expansion of π_{p+1} in powers of π_{p+n} has only even terms (mod 2), i.e. that $\pi_{p+1} \in A_{p+n}^2 + 2A_{p+n}$. This follows from the Eisenstein equation 4.11. Indeed, $\pi_k \equiv \pi_{k+1}^2+a_k \pmod 2$ and

$$a_k = \frac{2-\zeta_{2^k}-\zeta_{2^k}^{-1}}{(\theta+\bar{\theta})^2} = \frac{(\zeta_4\zeta_{2^{k+1}}+\zeta_4^{-1}\zeta_{2^{k+1}}^{-1})^2}{(\theta+\bar{\theta})^2} \in A_{k+1}^2$$

so that $\pi_k \in A_{k+1}^2 + 2A_{k+1}$ and especially $\pi_{p+1} \in A_{p+2}^2+2A_{p+2} \subset A_{p+n}^2+2A_{p+n}$. //

The calculation of $\varphi: H^0(A[\mathbb{Z}/2^{k-1}]^{\times}) \to H^0(A(k)^{\times})$ in case 4.4(i) is quite similar. We summarize our calculations in

<u>Proposition 4.23</u>. Let the residue field $\mathbb{F}$ of $A = A_0 = \tilde{A}_0$ have order 2^f. Consider $C_A(k)$, $A[\mathbb{Z}/2^{k-1}]$ with $\beta = 1$. The maps

$$\varphi(k): H^0(C_A(k)^\times) \to H^0(A(k)^\times),\ \varphi(k): H^0(A[\mathbb{Z}/2^{k-1}]^\times) \to H^0(A(k)^\times)$$

have the same rank. Their kernels and cokernels are elementary abelian 2-groups with rank 0 if $k=2$ and

$$rk(\operatorname{Ker}\varphi(3)) = f-1,\ rk(\operatorname{Ker}\varphi(k)) = (3\cdot 2^{k-4}-1)f-(k-3),\ k \geq 4$$

$$rk(\operatorname{Coker}\varphi(3)) = f,\quad rk(\operatorname{Coker}\varphi(k)) = (3\cdot 2^{k-4}-1)f+1,\quad k \geq 4.$$

More precisely, $\operatorname{Coker}\varphi(k) = \bigoplus_{n=2}^{k-1} Cok_n$ where $Cok_n = H^0(A_n^\times/U_n)$ with $U_2 = 1+2A_2^\times$, $U_3 = \{1+(c^2+c)\pi_3^3+2d \mid c,d \in A_3\}$ and $U_n = 1+\pi_n^m A_n^\times$, $m = 3\cdot 2^{n-3}$ for $n \geq 4$.

<u>Remark 4.24</u>. Choosing a system of representatives F for $\mathbb{F}$ in A_0 we find canonical representatives for Cok_n as in 4.19:

$$Cok_2 = \{1+c\pi \mid c \in F\}$$

$$Cok_3 = \{1+c_1\pi+c_3\pi^3 \mid c_1 \in F,\ c_3 \in F/p(F)\}$$

$$Cok_n = \{1+c_1\pi+c_3\pi^3+\ldots+c_{m-1}\pi^{m-1} \mid c_i \in F,\ m = 3\cdot 2^{n-3}\},\ n \geq 4.$$

where $p: F \to F$ is the (unique) lift of $\bar{p}(x) = x^2+x$ on $\mathbb{F}$.

<u>Theorem 4.25</u>. In the situation of 4.4 with $\sigma_2 \neq 1$,

$$\bar{\psi}_1^O: g_2\cdot L_1^X(C,\beta,T_0) \to g_2\cdot \prod_{\text{type } O} H^0(A_1^\times)$$

has the following kernel and cokernel

(i)
$$\operatorname{Ker}\bar{\psi}_1^O = \begin{cases} g_2\cdot(\operatorname{Ker}\varphi(r-1)\times H^0(1+2\tilde{A}_{r-1})/(\mathbb{Z}/2\oplus\mathbb{Z}/2)), & r \geq 3 \\ g_2\cdot H^0(1+2\tilde{A}_1^\times)/(\mathbb{Z}/2) \cong g_2\cdot H^0(\tilde{A}_1^\times)/(\mathbb{Z}/2), & r = 2 \end{cases}$$

and $\operatorname{Coker}\bar{\psi}_1^O = g_2\cdot\operatorname{Coker}\varphi(r-1)$ for $r \geq 2$.

(ii) $\operatorname{Ker}\bar{\psi}_1^O = g_2\cdot H^0(1+2\tilde{A}_{r-1}^\times) = g_2\cdot\mathbb{F}$, $\operatorname{Coker}\bar{\psi}_1^O = 0$

(iii) $\operatorname{Ker}\bar{\psi}_1^O = g_2\cdot(\operatorname{Ker}\varphi(r-1)\times H^0(1+2A_{r-1}^\times)/(\mathbb{Z}/2\oplus\mathbb{Z}/2)$
$\operatorname{Coker}\bar{\psi}_1^O = g_2\cdot\operatorname{Coker}\varphi(r-1)$.

(iv) $\operatorname{Ker}\bar{\psi}_1^O = g_2\cdot H^0(1+2A_{r-1}^\times) \cong g_2\cdot\mathbb{F}$, $\operatorname{Coker}\bar{\psi}_1^O = 0$.

Here $\operatorname{Ker}\varphi(2) = \operatorname{Coker}\varphi(2) = 0$ and $\operatorname{Ker}\varphi(r-1)$, $\operatorname{Coker}\varphi(r-1)$ are listed in 4.24.

<u>Proof</u>. We have the exact sequence

$$L_2^K(C,\beta,T_0) \xrightarrow{d_2} H^2(C^\times) \to L_1^X(C,\beta,T_0) \to 0$$

and the image of d_2 is $\mathbb{Z}/2$ with generator $\langle 1+(1-T_0)^2/T_0 b\rangle$ where $\bar{b} \in \mathbb{F}$ is not of the form $x+x^2$, cf. 3.15. This element has image $1+4b$ in the single type Sp summand and has trivial image in the type O summands. Thus for case (i) and (iii) in 4.14. We divide out $\langle 1-4b\rangle$ in the top component $H^0(1+2\tilde{A}_{r-1}^\times)/\langle -1\rangle$ (resp. $H^0(1+2A_{r-1}^\times)/\langle 1+2\gamma\rangle$), in case (ii) and (iv) it is the $\mathbb{Z}/2$ which is divided out.

Now, (i) follows from the diagram

$$\begin{array}{ccccccccc}
1 \to & H^0(\tilde{A}_{r-1}^\times) & \to & H^0(A(r)^\times) & \to & H^0(A(r-1)^\times) & \to 1 \\
& \uparrow f & & \uparrow \varphi(r) & & \uparrow \varphi(r-1) & \\
1 \to & H^0(1+2\tilde{A}_{r-1}^\times) & \to & H^0(A[\mathbb{Z}/2^{r-1}]^\times) & \to & H^0(A[\mathbb{Z}/2^{r-1}]^\times) & \to 1
\end{array}$$

and similarly for case (iii). In case (ii), $\bar{\psi}_1^O$ is g_2 copies of

$$H^0(1+2\tilde{A}_{r-1}^\times)\times H^0(A_0^\times) \xrightarrow{\text{proj.}} H^0(A_0^\times)$$

and similarly in case (iv). This proves the result. //

§5. Intermediate L-groups: 2-hyperelementary case.

Let $\tau = \mathbb{Z}/n \tilde{\times} \sigma$ be as in 1.2, and let $d > 1$ be a divisor of n such that $t: \sigma \to (\mathbb{Z}/d)^{\times}$ has abelian kernel. We use the results of the previous paragraphs to evaluate (modulo extensions) the groups $L_i^X(\mathbb{Z}\tau)(d) = L_i^X(\mathbb{Z}\tau, \alpha, 1)(d)$.

Let $Y(R) = X(R) \oplus \langle -1 \rangle \oplus \tau/\tau'$. Since $Y/X = \langle -1 \rangle \oplus \tau/\tau'$ and $H^i(\tau/\tau') = H^i(\sigma/\sigma')$ we have from 2.1 and 2.8(i),

5.1 $$L_i^Y(\mathbb{Z}\tau) = \prod_{d>1} L_i^X(\mathbb{Z}\tau)(d) \times L_i^Y(\mathbb{Z}\sigma) .$$

The groups $L_i^Y(\mathbb{Z}\sigma)$ were calculated in [W6] for $\sigma = \mathbb{Z}/2^k$ and $\sigma = Q2^k$, so our results give $L_i^Y(\mathbb{Z}\tau)$ for the groups of type I and type IIA (in 1.2). Explicitly, these are the following groups:

5.2 $$\tau = \mathbb{Z}/n \tilde{\times} \mathbb{Z}/2^k; \; Q(2^k a); \; Q(8a,b,c), \; a \geq b \geq c$$

and products $\tau \times \mathbb{Z}/a_0$ with $(a_0, |\tau|) = 1$.

The intermediate L-groups $L_i'(\tau)$ (equal to $L_i^S(\tau)$ when $SK_1(\mathbb{Z}\tau) = 0$) are derived from $L_i^Y(\mathbb{Z}\tau)$:

$$L_{2i}'(\tau) = L_{2i}^Y(\mathbb{Z}\tau), \; L_{2i-1}'(\tau) = L_{2i-1}^Y(\mathbb{Z}\tau)/\langle A \rangle \quad \text{where } A = \begin{pmatrix} 0 & 1 \\ (-1)^i & 0 \end{pmatrix}.$$

Note for all groups in 5.2, $L_i'(\tau) = L_i^S(\tau)$ but $L_i'(\tau \times \mathbb{Z}/a_0) \neq L_i^S(\tau \times \mathbb{Z}/a_0)$ when $\tau \supseteq Q8$ by results from [O3].

Also note from [O3], that for all rings $R_p(d)$ we consider, $SK_1(R_p(d)) = 0$. Thus $L_i^X(R_p(d)) = L_i(R_p(d))$ for all $p \nmid d$.

We follow the outline presented at the end of §2 for evaluating the main exact sequence. We start by a calculation of

$$\gamma_n: \prod_{p \nmid 2d} L_n^X(R_p(d)) \oplus L_n(T(d)) \to CL_n(S(d))$$

and then insert the results from §4 about $L_n^X(R_2(d))$. The maps γ_n decomposes in accordance with the decomposotion of the anti-structure $(S(d),\alpha,1)$ into simple factors. The types of the individual factors will be important and we write γ_n^O, γ_n^{Sp} and γ_n^U when we wish to emphasize the type in question.

Let $(S,\alpha,1)$ a simple factor in $(S(d),\alpha,1)$ with center E. Then $S \otimes_{\mathbb{Q}} \mathbb{R} \subset T(d)$ breaks up into isomorphic components which are Morita equivalent to $(\mathbb{C},c,1)$, $(\mathbb{H},c,1)$ or $(\mathbb{R},1,\pm 1)$, where c is the standard (anti-)involution on $\mathbb{C}$ and $\mathbb{H}$.

First, suppose that $(S,\alpha,1)$ has type U. Let A be the integers of E. For each prime $p \nmid 2d$, $L_n(R_p(d)) = L_n(A_p) = 0$, because $(A_p,\alpha,1)$ is a direct sum of type U or type GL antistructures. We have $CL_i(S) = 0$ by 2.19(i), so $\mathrm{Coker}\gamma_n^U = 0$ and $\mathrm{Ker}\gamma_n^U = \bigoplus L_n(\mathbb{C},c,1)$, with one summand for each component in $S \otimes_{\mathbb{Q}} \mathbb{R}$. From 2.17(i), we get

5.3 $$\mathrm{Ker}\gamma_{2n}^U = \Sigma^U, \quad \mathrm{Ker}\gamma_{2n+1}^U = 0, \quad \mathrm{Coker}\gamma_n^U = 0$$

where $\Sigma^U = \bigoplus 4\mathbb{Z}$, summand over conjugate pairs of (complex) primes in E.

Second, suppose $(S,\alpha,1)$ has type O or type Sp. We shall need notation for different subproducts of $\hat{A} = \Pi A_p = \Pi\{A_y \mid y \in P(E)\}$, cf. 2.2. Set

5.4 $$A_m = \Pi\{A_p \mid p \in P_m(E)\}, \; A_{m'} = \hat{A}/A_m$$
$$E_\infty = \mathbb{R} \otimes_{\mathbb{Q}} E = \bigoplus \mathbb{R} \quad (|P_\infty(E)| \text{ summands})$$

We have $\alpha|E = 1$, so in the tables below the cohomology groups of $A_n^\times$ and $E_\infty^\times$ are products of

$$H^0(A_y^\times) = A_y^\times/(A_y^\times)^2, \; H^1(A_y^\times) = \langle -1 \rangle, \; H^i(\mathbb{R}^\times) = \langle -1 \rangle .$$

We give the calculation of $\gamma_n = \gamma_n^O$ when $(S,\alpha,1)$ has type O in 5.5. The first column is the source of γ_n^O, the second is the target. Notation as in §2

Table 5.5: type O.

n	$\Pi\{L_n(A_p) \mid p\nmid 2d\} \times L_i(E_\infty)$	$CL_n(E)$	$\mathrm{Ker}\gamma_n^O$	$\mathrm{Coker}\gamma_n^O$
0	$0 \times \oplus(4\mathbb{Z})$	$\mathbb{Z}/2$	Σ^O	0
1	$H^0(A_{2d'}^\times) \times H^0(E_\infty^\times)$	$H^0(C(E))$	$\mathrm{Ker}\gamma_1^O$	$\mathrm{Coker}\gamma_1^O$
2	$H^1(A_{2d'}^\times)\quad H^1(E_\infty^\times)$	$H^1(C(E))$	0	$H^1(A_{2d}^\times)/H^1(A^\times)$
3	0×0	0	0	0

The table follows from the results in 2.15 - 2.20 but a few extra comments are in order: γ_0^O maps each $4\mathbb{Z}$ onto $\mathbb{Z}/2$ and has kernel $\Sigma^O = 8\mathbb{Z} \oplus (\bigoplus 4\mathbb{Z})$ of rank $|P_\infty(E)|$. The maps γ_1^O, γ_2^O are the natural homomorphisms. The global square theorem gives

$$H^1(C(E)) \cong H^1(E_A^\times)/H^1(E^\times) \cong H^1(A^\times \times E_\infty^\times)/H^1(A^\times)$$

and the results for $\mathrm{Ker}\gamma_2^O$ and $\mathrm{Coker}\gamma_2^O$ follows.

Similarly, $H^0(C(E)) \cong H^0(E_A^\times)/H^0(E^\times)$, so γ_1^O has the same kernel and cokernel as

$$H^0(A_{2d'}^\times) \times H^0(E_\infty^\times) \times H^0(E^\times) \to H^0(E_A^\times).$$

We return to this after discussing type Sp summands, where we have

Table 5.6: type Sp.

n	$\Pi\{L_n(A_p) \mid p\nmid 2d\} \times L_n(\mathbb{H}(E_\infty))$	$CL_n(E)$	$\mathrm{Ker}\gamma_n^{Sp}$	$\mathrm{Coker}\gamma_n^{Sp}$
0	$H^1(A_{2d'}^\times) \times \oplus(2\mathbb{Z})$	$H^1(C(E))$	Σ^{Sp}	$H^1(A_{2d}^\times)/H^1(A^\times)$
1	0×0	0	0	0
2	0×0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
3	$H^0(A_{2d'}^\times) \times 0$	$H^0(C(E))$	$\mathrm{Ker}\gamma_3^{Sp}$	$\mathrm{Coker}\gamma_3^{Sp}$

Note that each $2\mathbb{Z}$ in the source of γ_0^{Sp} surjects onto the corresponding $\langle -1\rangle \in H^1(E_\infty^\times)$, so the kernel is $\Sigma^{Sp} = \bigoplus\{4\mathbb{Z} \mid P_\infty(E)\}$. Finally, γ_3^{Sp} has the same kernel and cokernel as

$$H^0(A_{2d'}^\times) \times H^0(E^\times) \to H^0(E_A^\times)$$

The maps γ^U, γ^O and γ^{Sp} are explicitly determined above except for the terms γ_1^O and γ_3^{Sp}. Now $E_A^\times/E_\infty^\times\hat{A}^\times = I(E)$, the ideal group of E, so the map

$$H^0(\hat{A}^\times) \times H^0(E_\infty^\times) \times H^0(E^\times) \to H^0(E_A^\times)$$

has the same kernel and cokernel as $H^0(E^\times) \to H^0(I(E))$. Hence the kernel is $E^{(2)}/E^{\times 2}$ where $E^{(2)} \subset E^\times$ consists of the elements with even valuation at all finite primes, and the cokernel is $H^0(\Gamma(E)) = \Gamma(E)/2\Gamma(E)$ where $\Gamma(E) = I(E)/E^\times$ is the ideal class group of E. The map γ_1^O (resp. γ_3^{Sp}) is obtained by deleting $H^0(A_{2d}^\times)$ (and $H^0(E_\infty^\times)$) from the source. A snake lemma argument gives the following exact sequences

5.7
$$0 \to \mathrm{Ker}\gamma_1^O \to E^{(2)}/E^{\times 2} \to H^0(A_{2d}^\times) \to \mathrm{Coker}\gamma_1^O \to H^0(\Gamma(E)) \to 0$$
$$0 \to \mathrm{Ker}\gamma_3^{Sp} \to E^{(2)}/E^{\times 2} \to H^0(A_{2d}^\times) \times H^0(E_\infty^\times) \to \mathrm{Coker}\gamma_3^{Sp} \to H^0(\Gamma(E)) \to 0$$

Alternatively, let $E^* \subset E^\times$ be the set of elements positive at all infinite primes. Then we may take $H^0(E_\infty^\times)$ away in the second sequence if we write $E^{*(2)} = E^* \cap E^{(2)}$ instead of $E^{(2)}$ and the strict class group $\Gamma(E)^* = I(E)/E^*$ instead of $\Gamma(E)$.

There is an exact sequence

5.8
$$0 \to H^0(A^\times) \to E^{(2)}/E^{\times 2} \to H^1(\Gamma(E)) \to 0$$

where $x \in E^{(2)}$ is mapped to $\frac{1}{2}\Sigma\, v_y(x) y \in \Gamma(E)$, and v_y denotes the valuation at y. Thus if $\Gamma(E)$ has odd order then in 5.7 $E^{(2)}/E^{\times 2}$ can be replaced with $H^0(A^\times)$ and $H^1(\Gamma(E))$ can be omitted. If further $\Gamma(E)^*$ has odd order then $H^0(A^\times)$ maps isomorphically to $H^0(E_\infty^\times)$, so $\mathrm{Ker}\gamma_3^{Sp} = 0$ and $\mathrm{Coker}\gamma_3^{Sp} = {}^0(A_{2d}^\times)$. In the exact sequence 2.21 the maps

$$\psi_n : \prod_{p \nmid d} L_n^X(R_p(d)) \times L_n(T(d)) \to CL_n(S(d))$$

differ from γ_n by the exact sequence

5.9 $\quad 0 \to \mathrm{Ker}\gamma_n \to \mathrm{Ker}\psi_n \to L_n^X(R_2(d)) \xrightarrow{\bar\psi_n} \mathrm{Coker}\gamma_n \to \mathrm{Coker}\psi_n \to 0.$

From §4 we have

$$L_n^X(R_2(d)) \cong g_2 L_n^X(C_y,\beta,T_0) \cong g_2 \cdot (H^{n+1}(C_y^\times)/\mathbb{Z}/2)$$

in all cases except if $\sigma_2 = 1$, which we treat separately later. We write C_y now instead of C to indicate dyadic completion.

We have $C_y \otimes Q = \prod_{i=0}^{r} E_{i,y}$, and precisely one factor has type Sp in $(C_y,\beta,T_0) \otimes Q$, cf. 4.4 and 4.14. Recall that $E_{0,y}$ and $E_{1,y}$ are umramified and that $E_{i,y}/E_{0,y}$ is totally ramified. The index $g_2 = ((\mathbb{Z}/d)^\times : \langle 2, t(\sigma)\rangle)$ is the number of dyadic primes in $E_0 = Q(\zeta_d)^\sigma$, and hence also the number of dyadic primes in any of the fields E_i.

Recall from 4.16 the maps

$$\bar\psi_n^O : L_n^X(R_2(d)) \to \prod_{\text{type O}} CL_n(E_i)$$
$$\bar\psi_n^{Sp} : L_n^X(R_2(d)) \to \prod_{\text{type Sp}} CL_n(E_i).$$

They were determined in 4.17 and 4.25.

Using 5.3, 5.5 and 5.6 (and a 'snake lemma' argument similar

to 5.7) we have an exact sequence

5.10
$$0 \to \mathrm{Ker}\bar\psi_1^O \to \mathrm{Ker}\psi_1 \to \prod_{\text{type O}} E_i^{(2)}/(E_i^\times)^2 \xrightarrow{\Pi\phi_i}$$
$$\to \prod_{\text{type O}} H^0(A_{i,d}^\times) \times \mathrm{Coker}\bar\psi_1^O \to \mathrm{Coker}\psi_1 \to \prod_{\text{type O}} H^0(\Gamma(E_i)) \to 0$$

Similarly, using that $\mathrm{Coker}\bar\psi_3^{Sp} = 0$ by 4.17 we get

5.11
$$0 \to \mathrm{Ker}\bar\psi_3^{Sp} \to \mathrm{Ker}\psi_3 \to \prod_{\text{type Sp}} E_i^{(2)}/(E_i^\times)^2 \xrightarrow{\Pi\phi_i^*}$$
$$\to \prod_{Sp}(H^0(A_{i,d}^\times) \times H^0(E_{i,\infty}^\times)) \to \mathrm{Coker}\psi_3 \to \prod_{\text{type Sp}} H^0(\Gamma(E_i)) \to 0$$

The products $\prod_{\text{type O}}$ (resp. $\prod_{\text{type Sp}}$) refer to the fields E_i for which the corresponding summand in $(S(d),\alpha,1)$ has type O (resp. type Sp). In 5.11, there is (at most) one factor in each product

The homomorphisms ψ_0, ψ_2 are easier. From 4.17 and 5.9 we get

5.12 $\qquad \mathrm{Ker}\psi_0 = \Sigma;\quad \mathrm{Coker}\psi_0 = \prod_{\text{type Sp}} H^1(A_{i,d}^\times)/H^1(A_i^\times)$

where $\Sigma = \Sigma^O \times \Sigma^{Sp} \times \Sigma^U$, and

5.13 $\mathrm{Ker}\psi_2 = \Sigma^U$, $\mathrm{Coker}\psi_2 = \prod_{\text{typeO}} H^1(A_{i,d}^\times)/H^1(A_i^\times) \times \prod_{\substack{i>0\\ \text{typeO}}} H^1(A_{i,2}^\times) \times \prod_{\text{typeSp}} (\mathbb{Z}/2)$

The rank of the groups $\mathrm{Ker}\bar\psi_1^O$, $\mathrm{Coker}\bar\psi_1^O$ were given in 4.25 and the rank of $\bar\psi_3^{Sp}$ in 4.17, thus we can calculate $L_n^X(\mathbb{Z}\tau)(d)$ from the exact sequence

$$0 \to \mathrm{Coker}\psi_{n+1} \to L_n^X(\mathbb{Z}\tau)(d) \to \mathrm{Ker}\psi_n \to 0$$

once we can determine the rank of $\Pi\phi_i$ and $\Pi\phi_i^*$ in 5.10 and 5.11.

We first make more explicit the exceptional case where $\tau = \mathbb{Z}/n \tilde\times \mathbb{Z}/2^k$ and $\sigma_2 = 1$.

Example 5.14. Let $\tau = \mathbb{Z}/n \tilde{\times} \mathbb{Z}/2^k$ and suppose $d|n$ is a divisor with $t\colon \mathbb{Z}/2^k \to (\mathbb{Z}/d)^\times$ injective and $-1 \in \operatorname{Image}(t)$. Let $E = Q(\zeta_d)^\sigma$, $A = \mathbb{Z}[\zeta_d]^\sigma$ where $\sigma = \mathbb{Z}/2^k$. The group $L_n^X(R_2(d))$ was listed in 4.6(ii). Especially $L_1^X(R_2(d)) \cong H^0(A_2^\times)$ and $L_2^X(R_2(d)) = \mathbb{Z}/2\langle c\rangle \oplus H^1(A_2^\times)$, and $\bar{\psi}_1^O$ is injective whereas $\bar{\psi}_2^O$ has kernel $\mathbb{Z}/2\langle c\rangle$. We get from 5.9-5.11

$$\operatorname{Ker}\psi_0 = 8\mathbb{Z} \oplus (4\mathbb{Z})^{m-1},\ m = \varphi(d)/2^k;\ \operatorname{Coker}\psi_0 = 0$$

$$0 \to \operatorname{Ker}\psi_1 \to E^{(2)}/E^{\times 2} \xrightarrow{\Phi} H^0(A_d^\times) \to \operatorname{Coker}\psi_1 \to \Gamma(E) \to 0$$

$$\operatorname{Ker}\psi_2 = g_2\cdot(\mathbb{Z}/2\langle c\rangle);\quad \operatorname{Coker}\psi_2 = H^1(A_d^\times)/H^1(A^\times)$$

$$\operatorname{Ker}\psi_3 = (H^0(A_2^\times)/g_2\cdot(\mathbb{Z}/2 \oplus \mathbb{Z}/2)) \oplus g_2\cdot(\mathbb{Z}/4);\ \operatorname{Coker}\psi_3 = 0,$$

where $g_2 = ((\mathbb{Z}/d)^\times : \langle 2, t(\mathbb{Z}/2^k)\rangle)$.

The g_2 copies of $\mathbb{Z}/4$ in $\operatorname{Ker}\psi_3$ are generated by the g_2 automorphisms $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus, when $t\colon \mathbb{Z}/2^k \to (\mathbb{Z}/d)^\times$ is injective,

$$L_1'(\tau) = L_1^Y(\mathbb{Z}\tau)/\mathbb{Z}/2,\quad L_3'(\tau) = L_3^Y(\mathbb{Z}\tau)/(\mathbb{Z}/4).$$

For all other 2-hyperelementary groups one only cancels a $\mathbb{Z}/2$ in each group $L_{2i+1}^Y(\mathbb{Z}\tau)$.

If $\tau_n = \mathbb{Z}/n \tilde{\times} \sigma$ and $d > 1$ a divisor of n then

$$L_n^X(\mathbb{Z}\tau_n)(d) = L_n^X(\mathbb{Z}\tau_d)(d)$$

where $\tau_d = \mathbb{Z}/d \tilde{\times} \sigma$ specified by $t\colon \sigma \to (\mathbb{Z}/n)^\times \to (\mathbb{Z}/d)^\times$. Thus in calculating $L_n^X(\mathbb{Z}\tau)(d)$ we need only consider the top component. We now list our results one by one for the 2-hyperelementary groups with periodic cohomology.

We take advantage of 3.4 and its corollaries and treat only the groups where -1 belongs to the image of $t\colon \sigma \to (\mathbb{Z}/d)^\times$.

Consider first $\sigma = \mathbb{Z}/2^k$ and $t(\sigma) = H$ of order 2^ℓ with $0 < \ell < k$. We are in case 4.4(ii),

$$(R_2(d), \alpha, 1) \sim g_2(E)\cdot(A[\mathbb{Z}/2^{k-\ell}], 1, T_0)$$

where $E = Q(\zeta_d)^H$ and $A = \mathbb{Z}[\zeta_d]^H$.
There is one type O factor and one type Sp factor, the other $k-\ell-1$ factors have type U.

We have the 2-ranks

$$\operatorname{rk}_2(\operatorname{Ker}\bar{\psi}_1^O) = |E:Q|,\ \operatorname{rk}_2(\operatorname{Ker}\bar{\psi}_3^{Sp}) = |E:Q|$$

$$\operatorname{rk}_2(\operatorname{Coker}\psi_0) = g_d(E)-1,\ \operatorname{rk}_2(\operatorname{Coker}\psi_2) = g_d(E)$$

where $g_d(E)$ is the number of primes in E dividing d.

Example 5.15. Let $\tau = \mathbb{Z}/d \tilde{\times} \mathbb{Z}/2^k$ and suppose $t\colon \mathbb{Z}/2^k \to (\mathbb{Z}/d)^\times$ has image H of order $2^\ell\,(0 < \ell < k)$ and $-1 \in H$. Then $L_i^X(\mathbb{Z}\tau)(d)$ is an extension of M_i by N_i in

i	M_i	N_i
0	$(\operatorname{Coker}\Phi) \oplus H^0(\Gamma(E))$	Σ
1	$g_d(E)\cdot(\mathbb{Z}/2)$	$m\cdot(\mathbb{Z}/2) \oplus \operatorname{Ker}\Phi$
2	$\operatorname{Coker}\Phi^* \oplus H^0(\Gamma(E))$	Σ^U
3	$(g_d(E)-1)\cdot\mathbb{Z}/2$	$m\cdot\mathbb{Z}/2 \oplus \operatorname{Ker}\Phi^*$

where $m = \varphi(d)/2^\ell$ and $\Phi\colon E^{(2)}/(E^\times)^2 \to H^0(A_d^\times)$, $\Phi^*\colon E^{(2)}/(E^\times)^2 \to H^0(A_d^\times) \oplus H^0(E_\infty^\times)$ are the natural maps for $E = Q(\zeta_d)^H$.

The free parts are $\Sigma = (8\mathbb{Z}) \oplus (4\mathbb{Z})^{m+m'-1}$ and $\Sigma^U = (4\mathbb{Z})^{m'-m}$, $m' = 2^{k-2\ell-1}\varphi(d)$.

If $\tau = Q(2^{k+1}d)$ we are in case 4.4(i) or 4.4(iii);

$$(R_2(d),\alpha,1) \sim g_i \begin{cases} (C_A(k),1,T^{2^{k-2}}), & -1 \in \langle 2\rangle \\ (A[\mathbb{Z}/2^{k-1}],1,T^{2^{k-2}}), & -1 \notin \langle 2\rangle \end{cases}$$

where $A = \mathbb{Z}_2[\zeta_d+\zeta_d^{-1}]$ and g_2 is the number of dyadic primes in $E = Q(\zeta_d+\zeta_d^{-1})$. The center of $S(d)$ is $\prod_{i=0}^{k-1} E_i$ where $E_i = Q(\zeta_{2^i s}+\zeta_{2^i d}^{-1})$.

<u>Example 5.16.</u> Let $\tau = Q(2^k d)$. We have from 4.17 and 4.25,

$$rk_2(\mathrm{Ker}\bar{\psi}_3^{Sp}) = 2^{k-3}\varphi(d)$$

$$rk_2(\mathrm{Ker}\bar{\psi}_1^{O}) = \begin{cases} (2^{k-2}+2^{k-4}+2^{k-5}-1)\varphi(d)/2 - (k-3)g_2, & k>4 \\ 5\varphi(d)/2 - 2g_2, & k=4 \\ \varphi(d) - g_2, & k=3 \end{cases}$$

$$rk_2(\mathrm{Coker}\bar{\psi}_1^{O}) = \begin{cases} (2^{k-4}+2^{k-5}-1)\varphi(d)/2 + g_2, & k>4 \\ \varphi(d)/2, & k=4 \\ 0, & k=3 \end{cases}$$

The group $L_i^X(\mathbb{Z}\tau)(d)$ is an extension of M_i by N_i in

	M_i	N_i
0	$\prod_{i=0}^{k-2}(H^0(\Gamma(E_i)) \times \mathrm{Coker}\Phi_i)$	Σ
1	$\prod_{i=0}^{k-2} g_{2d}(E_i)+k-g_2(E_0))\cdot\mathbb{Z}/2$	$\mathrm{Ker}\bar{\psi}_1^{O} \times \prod_{i=0}^{k-2}(\mathrm{Ker}\Phi_i)$
2	$\mathrm{Coker}\ \Phi_{k-1}^{*} \times \Gamma(E_{k-1})$	0
3	$(g_d(E_{k-1})-1)\cdot\mathbb{Z}/2$	$\mathrm{Ker}\Phi_{k-1}^{*} \times (2^{k-3}\varphi(d))\cdot\mathbb{Z}/2$

Here $g_d(E_i) = \Sigma g_p(E_i)$ is the number of primes in E_i dividing d, and $g_2 = g_2(E_i) = g_2(E_0)$. The torsion free part Σ is $\Sigma = (8\mathbb{Z})^{k-1} \oplus (4\mathbb{Z})^m$, $m = 2^{k-2}\varphi(d) - (k-1)$. Finally,

$$\Phi_i\colon E_i^{(2)}/(E_i^{\times})^2 \to H^0(A_{i,d}^{\times}) \times \mathrm{Cok}_i$$

$$\Phi_{k-1}^{*}\colon E_{k-1}^{(2)}/(E_{k-1}^{\times})^2 \to H^0(A_{k-1,d}^{\times}) \times H^0(E_{k-1,\infty}^{\times})$$

are the natural maps, and $\mathrm{Cok}_i = 0$ for $i \le 2$ but for $i \ge 3$ $\mathrm{Cok}_i = g_2(E_i)\cdot H^0(A_{i,y}^{\times}/U_i)$ with $U_3 = \{1+(c^2+c)\pi_3^3+2d \mid c,d \in A_{3,y}\}$ and $U_i = 1+\pi_i^m A_{i,y}$ with $m = 2^{i-2}+2^{i-3}$ for $i \ge 4$ ($A_{i,y}$ denotes a dyadic completion).

For the groups $\tau = Q(2^{k+1}a,b,c)$ and $d = abc$. the center of $S(d)$ consists of totally complex fields (cf. 1.14) and 3.6 gives

5.17 $\qquad L_{2n+1}^X(\mathbb{Z}\tau)(d) = 0$ and $L_{2n}^X(\mathbb{Z}\tau)(d) = (4\mathbb{Z})^m$, $m = 2^{k-5}\varphi(d)$.

if all three numbers are different from 1. Thus we are left with two non-trivial cases

5.18 $\qquad \tau = Q(2^{k+1}a,b,1)$ and $\tau = Q(2^{k+1},b,c)$

(Note that $Q(2^{k+1},b,1)$ falls outside the scope of this paper, when $k>2$ since $t\colon Q2^{k+1} \to (\mathbb{Z}/b)^{\times}$ has then non-abelian kernel. It is handled in the forthcoming part II).

If $k=3$, then the two cases in 5.18 coincide, and we are in case 4.4(i):

$$(R_2(d),\alpha,1) \sim (A_2[\mathbb{Z}/2],1,T_0)$$

where $A = \mathbb{Z}[\zeta_a+\zeta_a^{-1}, \zeta_b+\zeta_b^{-1}]$ and $T_0 \in \mathbb{Z}/2$ is the generator.

The center of $S(d)$ is $E \times E$, $E = \mathbb{Q}(\zeta_a+\zeta_a^{-1}, \zeta_b+\zeta_b^{-1})$.

Example 5.19. Let $\tau = Q(8a,b,1)$, $d = ab$. If both a and b are larger than 1 then $L_n^X(\mathbb{Z}\tau)(d)$ is given in 5.15 with $E = \mathbb{Q}(\zeta_a+\zeta_a^{-1}, \zeta_b+\zeta_b^{-1})$, $m = \varphi(d)/4$, $\Sigma^U = 0$ and $\Sigma = 8\mathbb{Z} \oplus (4\mathbb{Z})^{2m-1}$. If $a=1$ or $b=1$ then the group is given in 5.16 with $k=3$.

For $\tau = Q(2^{k+1}a,b,1)$, we are in case 4.4(i) or (iii):

$$(R_2(d),\alpha,1) \sim g_2 \cdot \begin{cases} (A[\mathbb{Z}/2^{k-1}],1,T^{2^{k-2}}) & \text{(case 4.4(i))} \\ (C_A(k),1,T^{2^{k-2}}) & \text{(case 4.4(iii))} \end{cases}$$

where case 4.4(i) occurs if and only if 2 is totally split in $\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_a+\zeta_a^{-1}, \zeta_b+\zeta_b^{-1})$. Here A is a dyadic completion of $\mathbb{Z}[\zeta_a+\zeta_a^{-1}, \zeta_b+\zeta_b^{-1}]$.

The center of $S(d)$ is the product of the fields

$$K_i = \mathbb{Q}(\zeta_{2^i a}+\zeta_{2^i a}^{-1}, \zeta_b+\zeta_b^{-1}), \quad i = 0,1,\dots,k-1.$$

Example 5.20. For $\tau = Q(2^{k+1}a,b,1)$, $k>2$ and $d = ab$, $L_n^X(\mathbb{Z}\tau)(d)$ is given by the table in 5.16 with E_i replaced by K_i, $i=0,\dots,k-1$.

Finally, for $\tau = Q(2^{k+1};b,c)$ and $d = bc$ we have case 4.4 (ii) or 4.4(iv):

$$(R_2(d),\alpha,1) \sim g_2 \cdot \begin{cases} (A[\mathbb{Z}/2^{k-1}],\beta,T_0), & \text{(case 4.4(ii)) or} \\ (C_A(k),\beta,T_0), & \text{(case 4.4(iv))} \end{cases}$$

(Case (ii) occurs if and only if 2 is totally split in

$\mathbb{Q}(\zeta_d)/\mathbb{Q}(\zeta_b+\zeta_b^{-1}, \zeta_c+\zeta_c^{-1})$. Here A is a dyadic completion of $\mathbb{Z}[\zeta_b+\zeta_b^{-1}, \zeta_c+\zeta_c^{-1}]$. The center of $S(d)$ is $\prod_{i=1}^{k-1} E_i$ with E_i given in 1.14(a) (with $a=1$).

Example 5.21. For $\tau = Q(2^{k+1},b,c)$, $k>2$ and $d=bc$, $L_n^X(\mathbb{Z}\tau)(d)$ is given by table 5.15 with $m = \varphi(d)/4$, $E = \mathbb{Q}(\zeta_b+\zeta_b^{-1}, \zeta_c+\zeta_c^{-1})$ and $m' = 2^{k-4}\cdot\varphi(d)$.

In the above we have only listed the components with $d>1$. We close the paragraph by listing the $L_n^Y(\mathbb{Z}\sigma) = L_n^Y(\mathbb{Z}\tau)(1)$. The result can be found in [W6].

5.22

$L_n^Y(\mathbb{Z}[\mathbb{Z}/2^k]) = (4\mathbb{Z})^m \oplus (8\mathbb{Z})^2;\ \mathbb{Z}/2;\ \mathbb{Z}/2 \oplus (4\mathbb{Z})^m;\ \mathbb{Z}/2 \oplus \mathbb{Z}/2$ for $n = 0,1,2,3$ where $m = 2^{k-1}-1$

$L_n^Y(\mathbb{Z}[Q2^k]) = (8\mathbb{Z})^{k+1} \oplus (4\mathbb{Z})^{m-1};\ \mathbb{Z}/2 \tilde{\times} m\cdot(\mathbb{Z}/2);\ \mathbb{Z}/2;\ (2^{k-3}+2)\cdot\mathbb{Z}/2$ for $n = 0,1,2,3$ and $m = 2^{k-2}-k+3$.

§6. L-theory of periodic groups.

In 1.19 we listed all groups π with periodic cohomology whose 2-hyperelementary subgroups $\tau = \mathbb{Z}/n \tilde{\times} \sigma$ have abelian kernel. In the previous paragraph we evaluated $L_n^Y(\mathbb{Z}\tau)$. It remains to recover $L_n^Y(\mathbb{Z}\pi)$ from Dress' induction theorem:

6.1 $$L_n^Y(\mathbb{Z}\pi) = \varprojlim L_i^Y(\mathbb{Z}\tau)\ .$$

Here $\varprojlim$ is taken over the category of 2-hyperelementary subgroups of π. The elements of $\varprojlim L^Y$ are sometimes called the stable elements (cf. [CE], [D]).

The free part is easy. We use the following notation for the irreducible characters χ of π

$$r_{\mathbb{R}}^U = \# \text{ conjugate pairs } (\chi,\bar{\chi}) \text{ with } \chi \neq \bar{\chi}$$

$$r_{\mathbb{R}}^O \,(\text{resp. } r_{\mathbb{R}}^{Sp}) = \# \text{ real valued } \chi, \Sigma\chi(g^2) = +|\pi| \,(\text{resp. } -|\pi|)$$

$$r_{\mathbb{Q}}^O = \# \text{ orbits of } Gal(Q(\zeta_{|\pi|})/Q) \text{ on type O real } \chi.$$

Proposition 6.2. Let π be a group with periodic cohomology. Then

$$L_0^Y(\mathbb{Z}\pi)/\text{Torsion} = \Sigma \oplus \Sigma^U, \quad L_2^Y(\mathbb{Z}\pi)/\text{Torsion} = \Sigma^U$$

where $\Sigma = (r_{\mathbb{R}}^O + r_{\mathbb{R}}^{Sp} - r_{\mathbb{Q}}^O)\cdot(4\mathbb{Z}) \oplus r_{\mathbb{Q}}^O\cdot(8\mathbb{Z})$ and $\Sigma^U = r_{\mathbb{R}}^U\cdot(4\mathbb{Z})$.

Proof. It is wellknown that the numbers $r_{\mathbb{R}}^O$, $r_{\mathbb{R}}^{Sp}$ etc. can be alternatively described as the number of type O, type Sp summands in $\mathbb{R}$ (cf. §3). We have checked 6.2 for 2-hyperelementary groups (which in the case of periodic groups include the p-hyperelementary groups). Both real and rational representation

theory satisfy 6.1, so the result follows. //

We have left to consider the torsion subgroups and go through the list 1.19 of periodic groups one by one. A type I group is a split extension

$$1 \to \mathbb{Z}/m \to \pi \to \mathbb{Z}/n \to 1$$

with presentation $\langle A,B \mid A^m = B^n = 1,\ BAB^{-1} = A^r\rangle$. The relatively prime numbers m and n become uniquely determined when we require that each Sylow subgroup of $\mathbb{Z}/n$ acts non-trivially on $\mathbb{Z}/m$. Then the Sylow 2-subgroup $\mathbb{Z}/2^k$ lies either in $\mathbb{Z}/m$, and $\tau = \mathbb{Z}/m$ is maximal 2-hyperelementary, or $\mathbb{Z}/2^k \subset \mathbb{Z}/n$ and $\tau = \mathbb{Z}/m \tilde{\times} \mathbb{Z}/2^k$ is the maximal one. In the latter case, recall that factors $d|m$ with $-1 \notin Im(\mathbb{Z}/2^k \to \mathbb{Z}/d^\times)$ do not contribute to the torsion (cf. 3.6).

We choose a maximal subgroup $\mathbb{Z}/\ell \subset \mathbb{Z}/m$ satisfying

$$-1 \in Im(\mathbb{Z}/2^k \to (\mathbb{Z}/\ell)^\times)$$

(Then $\tau = \mathbb{Z}/m \tilde{\times} \mathbb{Z}/2^k$ has the same L^Y-torsion as $\mathbb{Z}/\ell \tilde{\times} \mathbb{Z}/2^k$).

Theorem 6.3. Let $\pi = \mathbb{Z}/m \tilde{\times} \mathbb{Z}/n$ be a periodic group of type I, $\mathbb{Z}/2^k$ the Sylow 2-subgroup, $k \geq 1$.

i) If $\mathbb{Z}/2^k \subset \mathbb{Z}/m$ then

$$\text{Tor } L_i^Y(\mathbb{Z}\pi) = 0;0;\mathbb{Z}/2;\mathbb{Z}/2 \text{ for } i = 0,1,2,3.$$

ii) If $\mathbb{Z}/2^k \subset \mathbb{Z}/n$ and $\mathbb{Z}/\ell \subset \mathbb{Z}/m$ is as above then

$$\text{Tor } L_i^Y(\mathbb{Z}\pi) = \text{Tor } L_i^Y(\mathbb{Z}[\mathbb{Z}/\ell \tilde{\times} \mathbb{Z}/2^k])^{\mathbb{Z}/n} =$$

$$= \Pi\, T_i(d), \quad d|\ell$$

iii) For each $d > 1$, $T_i(d)$ is obtained from 5.14 by replacing the field E with $E^{\mathbb{Z}/n}$, and for $d = 1$, $T_i(1) = \operatorname{Tor} L_i^Y(\mathbb{Z}[\mathbb{Z}/2^k])$ is given in 5.22.

<u>Proof</u>. i) If $\mathbb{Z}/2^k \subset \mathbb{Z}/m$ then $\tau = \mathbb{Z}/m$ is a maximal hyperelementary subgroup. Hence $\operatorname{Tor} L_i^Y(\mathbb{Z}\pi) = \operatorname{Tor} L_i^Y(\mathbb{Z}\tau)^{\mathbb{Z}/n} = \operatorname{Tor} L_i^Y(\mathbb{Z}[\mathbb{Z}/2^k])^{\mathbb{Z}/n}$ and the result follows from 5.22, since the two $\mathbb{Z}/2$ are clearly stable.

ii) If $\mathbb{Z}/2 \subset \mathbb{Z}/n$ then $\tau = \mathbb{Z}/\ell \tilde{\times} \mathbb{Z}/2^k$ is a maximal hyperelementary group as far as $\operatorname{Tor} L_i^Y$ is concerned. We have to study the action of conjugation on subgroups of τ. The elements of $\mathbb{Z}/n$ give automorphisms on the whole τ. Conjugation c by an element $A^s \in \mathbb{Z}/m$ induces identity on $\mathbb{Z}/\ell$ and

$$(*) \qquad A^sB^tA^{-s} = A^{s(1-r^t)}B^t$$

Let $\tau = \langle A^p, B^q\rangle$ and write $G \subseteq \mathbb{Z}/2^k$ for the maximal subgroup with $c(G) \subset \tau$. Then c induces an automorphism of $\tau' = \mathbb{Z}/\ell \tilde{\times} G$ and the claim follows if we can show that c induces identity on $\operatorname{Tor} L_i^Y(\mathbb{Z}\tau')$.

The calculation of $L_i^Y(\mathbb{Z}\tau')$ was reduced to the centers of $Q\tau'$, $\mathbb{Z}_p\tau'$ and type U summands gave no torsion. It thus suffices to show that c is the identity on the centers of type O and Sp summands. For $Q\tau'$ and $\mathbb{Z}_p\tau'$, p odd, this is clear since they are subrings of $Q(\zeta_\ell)$ and $\mathbb{Z}_p[\zeta_\ell]$ and $c(\zeta_\ell) = \zeta_\ell$. The center of $\mathbb{Z}_2[\zeta_\ell]^t[G]$ is AH where $A = \mathbb{Z}_2[\zeta_\ell]^G$ and $H = \operatorname{Ker}(G \to (\mathbb{Z}/\ell)^\times)$. The subgroup $H \subset \tau'$ is characteristic as the Sylow 2-subgroup of the center of τ', so c induces an automorphism of H. But c has odd order and $\operatorname{Aut}(H)$ is a cyclic 2-group. Hence $c = \mathrm{id}$.

iii) The groups $\operatorname{Tor} L_i^X(\mathbb{Z}\tau)(d)$ lie in exact sequences of elementary abelian 2-groups. They are cohomologically trivial $\mathbb{Z}/n'$-modules, where $n' = n/2^k$ is odd. Hence we can take invariants termwise. This corresponds to taking Galois invariants of the number fields and rings involved. //

In the remaining cases of 1.19, the 2-hyperelementary subgroups are products $\mathbb{Z}/a_0 \times Q(2^ka,b,c)$ where either $k = 3$ or $k > 3$ and $b = c = 1$. We can disregard the factor $\mathbb{Z}/a_0$ and even more by 5.17 we are left with only two cases: $\tau = Q(8a,b,1)$ or $\tau = Q(2^ka)$.

For groups of type II, cf. 1.19(ii) we have by a similar argument,

<u>Proposition 6.4</u>. Suppose $\pi = \mathbb{Z}/m \tilde{\times} (\mathbb{Z}/n \times Q2^k)$ is such that either $\mathbb{Z}/m \tilde{\times} Q2^k = \mathbb{Z}/a_0 \times Q(2^ka)$ or $\mathbb{Z}/m \tilde{\times} Q2^k = \mathbb{Z}/a_0 \times Q(8a,b,c)$. In the first case,

$$\operatorname{Tor} L_i^Y(\mathbb{Z}\pi) = \operatorname{Tor} L_i^Y(\mathbb{Z}[Q(2^ka)])^{\mathbb{Z}/n} = \prod_{d|\ell} T_i(d)$$

For $d > 1$, $T_i(d)$ is obtained from the table in 5.16 by disregarding the torsion free Σ and replacing E_i with $E_i^{\mathbb{Z}/n}$; $T_i(1) = \operatorname{Tor} L_i^Y(\mathbb{Z}[Q2^k])$ is given in 5.21. In the second case $(k = 3)$,

$$\operatorname{Tor} L_i^Y(\mathbb{Z}\pi) = \Pi\{T_i(d) \mid (d,a),(d,b) \text{ or } (d,c) \text{ is equal to } 1\}.$$

and $T_i(d)$ is the $\mathbb{Z}/n$-invariant part of 5.19.

The remaining groups with periodic cohomology whose 2-hyperelementary subgroups have abelian kernel can be tabulated as follows

III. $1 \to \pi_1 \times Q8 \to \pi \to \mathbb{Z}/3 \to 1$, $Q8 \tilde{\times} \mathbb{Z}/3 = S\ell_2(\mathbb{F}_3)$

IV. $\pi_1 \times T\ell_2(\mathbb{F}_3)$

V. $\pi_1 \times S\ell_2(\mathbb{F}_p)$, $p \geq 5$

VI. $\pi_1 \times T\ell_2(\mathbb{F}_p)$, $p \geq 5$.

It follows from 3.6 and 6.1 that we have

Lemma 6.5. (i) $\mathrm{Tor}\, L_n^Y(\mathbb{Z}\pi) \cong \mathrm{Tor}\, L_n^Y(\mathbb{Z}[S\ell_2(\mathbb{F}_p)])$ (III,V)
(ii) $\mathrm{Tor}\, L_n^Y(\mathbb{Z}\pi) \cong \mathrm{Tor}\, L_n^Y(\mathbb{Z}[T\ell_2(\mathbb{F}_p)])$ (IV,VI).

Finally, to calculate $L_n^Y(\mathbb{Z}\pi)$ (for $\pi = S\ell_2(\mathbb{F}_p), T\ell_2(\mathbb{F}_p)$) we need to know the image of $L_n^Y(\mathbb{Z}\pi) \to L_n^Y(\mathbb{Z}\pi_2)$, where $\pi_2 = Q2^k \in Syl_2(\pi)$.

Let T_v^* denote the semi-direct product $Q8 \tilde{\times} \mathbb{Z}/3^v$, where $t\colon \mathbb{Z}/3^v \to \mathrm{Aut}(Q8)$ maps the generator onto the cyclic permutation of X, Y and XY. For $v = 1$, $T_v^* = S\ell_2(\mathbb{F}_3)$.

The argument in [M, Lemma 1.10] for generalized cohomology theories applies to characterize the image of $i^*\colon L^Y(\mathbb{Z}\pi) \to L_n^Y(\mathbb{Z}[Q2^k])$ as follows

Lemma 6.6. Let π have periodic cohomology and suppose $Q2^k \in Syl_2(\pi)$. An element $x \in L_n^Y(\mathbb{Z}[Q2^k])$ comes from $L_n^Y(\mathbb{Z}\pi)$ if and only if for each $Q8 \subset Q2^k$, contained in a subgroup $T_v^* \subset \pi$, the restriction of x to $L_n^Y(\mathbb{Z}[Q8])$ extends to $L_n^Y(\mathbb{Z}[T_v^*])$.

Given $Q8 \subset T_v^*$, we have from 6.1

$$L_n^Y(\mathbb{Z}[Q8])^{\mathbb{Z}/3} = \mathrm{Image}(L_n^Y(\mathbb{Z}[T_v^*]) \to L_n^Y(\mathbb{Z}[Q8])).$$

Thus to apply 6.5, we need to determine this invariant part and the two restrictions

$$\mathrm{Res}_j\colon L_n^Y(\mathbb{Z}[Q2^k]) \to L_n^Y(\mathbb{Z}[Q8]).$$

Corresponding to the two conjugacy classes of subgroups $Q8$ in $Q2^k$.

The group $L_0^Y(\mathbb{Z}[Q2^k])$ is torsion free, and information about its stable part is contained in 6.2. We have $L_2^Y(\mathbb{Z}[Q2^k]) = \mathbb{Z}/2$; its image under Res_j lifts to any $L_2^Y(\mathbb{Z}[T_v^*])$. For the odd groups, we prove in the appendix:

Theorem 6.7. (i) $L_1^Y(\mathbb{Z}[Q2^k])$ is an extension of $\mathbb{Z}/2$ by $(2^{k-2} - k+3)\cdot\mathbb{Z}/2$ and $L_3^Y(\mathbb{Z}[Q2^k]) = (2^{k-3}+2)\cdot\mathbb{Z}/2$.

(ii) $L_1^Y(\mathbb{Z}[Q8])^{\mathbb{Z}/3} = L_1^Y(\mathbb{Z}[Q8])$, $L_3^Y(\mathbb{Z}[Q8])^{\mathbb{Z}/3} = \mathbb{Z}/2\langle\tau\rangle$; the invariant element τ does not lift to $L_3^Y(\mathbb{Z}[Q2^k])$, $k > 3$.

(iii) The two restrictions $\mathrm{Res}_j\colon L_3^Y(\mathbb{Z}[Q2^k]) \to L_3^Y(\mathbb{Z}[Q8])$ have rank 2 for $k = 4$ and rank 1 for $k > 4$. Moreover, $\mathrm{Ker}(\mathrm{Res}_1) \neq \mathrm{Ker}(\mathrm{Res}_2)$.

We begin with $S\ell_2(\mathbb{F}_p)$. It is well-known that it has three basic hyperelementary subgroups arising as normalizer of elements. The first is an extension $\mathbb{F}_p \tilde{\times} \mathbb{Z}/2^\ell$ where $\mathbb{Z}/2^\ell \in Syl_2(\mathbb{F}_p^\times)$ with the square of the natural action and the others are quaternion groups $\tau_2 = Q(2(p-1))$, $\tau_3 = Q(2(p+1))$. It is a (basic) fact that the groups τ_1, τ_2 and τ_3 intersect only at the Sylow 2-subgroup. The groups τ_2 and τ_3 are selfnormalizing and the normalizer of τ_1 is $\mathbb{F}_p \tilde{\times} \mathbb{F}_p^\times$, giving $\mathbb{F}_p^\times/(\mathbb{Z}/2^\ell)$ as a group of automorphisms.

Each $Q8 \subset Q2^k$ is contained in a $T_1^* \subset S\ell_2(\mathbb{F}_p)$, so induction gives

6.8 $L_n^Y(\mathbb{Z}[S\ell_2(\mathbb{F}_p)]) = L_n^Y(\mathbb{Z}[Q2^k])^{st} \oplus L_n^Y(\mathbb{Z}\tau_1)_{od}^{\mathbb{F}_p^\times} \oplus L_n^Y(\mathbb{Z}\tau_2) \oplus L_n^Y(\mathbb{Z}\tau_3)$

Here $L_n^Y(\mathbb{Z}[Q2^k])^{st}$ can be calculated from 6.6 and 6.7:

$$L_1^Y(\mathbb{Z}[Q2^k])^{st} = L_1^Y(\mathbb{Z}[Q2^k]) = \mathbb{Z}/2 \tilde{\times} [(2^{k-2}-k+3)\mathbb{Z}/2]$$

6.9 $$L_3^Y(\mathbb{Z}[Q2^k])^{st} = 2^{k-3}(\mathbb{Z}/2), \quad k \geq 5$$

$$= \mathbb{Z}/2, \qquad k = 3,4.$$

where $(m\cdot\mathbb{Z}/2)\tilde{\times}(n\cdot\mathbb{Z}/2)$ means any extension $0 \to m(\mathbb{Z}/2) \to A \to n(\mathbb{Z}/2) \to 0$, and $L_n^Y(\mathbb{Z}\tau_1)_{od} = \prod_{d>1} L_n^Y(\mathbb{Z}\tau_1)(d)$.

Quite similarly, $T\ell_2(\mathbb{F}_p)$ has three basic 2-hyperelementary groups: τ_1: $\mathbb{F}_p \tilde{\times} \mathbb{Z}/2^{\ell+1}$ with the natural action of the quotient $\mathbb{Z}/2^{\ell+1} \to \mathbb{Z}/2^\ell \subset \mathbb{F}_p^\times$ and $\tau_2' = Q(4(p-1))$, $\tau_3' = Q(4(p+1))$. This time, however only one of the conjugacy classes $Q8 \subset Q2^{k+1}$ is contained in a T_v^*, the other is not, so $L_3^Y(\mathbb{Z}[Q2^{k+1}])^{st} \subset L_3^Y(\mathbb{Z}[T\ell_2(\mathbb{F}_p)])$ is given by

6.10 $$L_3^Y(\mathbb{Z}[Q2^{k+1}])^{st} = \begin{cases} (2^{k-2}+1)\cdot(\mathbb{Z}/2) & k > 3 \\ 2\cdot(\mathbb{Z}/2) & k = 3 \end{cases}$$

Replacing τ_i by τ_i' in 6.8 and using 6.10 instead og 6.9 we get a formula for $L_n^Y(\mathbb{Z}[T\ell_2(\mathbb{F}_p)])$.

<u>Theorem 6.11</u>. The torsion in $L_n^Y(\mathbb{Z}\tau_1)_{od}^{\mathbb{F}_p}$ is given by

n	$S\ell_2(\mathbb{F}_p)$			$T\ell_2(\mathbb{F}_p)$	
	$p \equiv 3(4)$	$p \equiv 5(8)$	$p \equiv 1(8)$	$p \equiv 3(4)$	$p \equiv 1(4)$
0	0	0	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$
1	0	$\mathbb{Z}/2\tilde{\times}(\mathbb{Z}/2)^3$	$\mathbb{Z}/2\tilde{\times}(\mathbb{Z}/2)^4$	$\mathbb{Z}/2\tilde{\times}\mathbb{Z}/2$	$\mathbb{Z}/2\tilde{\times}(\mathbb{Z}/2)^2$
2	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
3	0	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

<u>Proof</u>. Example 5.15 describes $L_i^Y(\mathbb{Z}\tau_1)_{od} = L_i^Y(\mathbb{Z}\tau_1)(p)$. Consider first $T\ell_2(\mathbb{F}_p)$. We have $E = Q(\zeta_p)^{\mathbb{Z}/2^\ell}$. As in 6.3iii) we take invariants by changing E to $E^{\mathbb{F}_p^\times} = Q$. Then $\Gamma(Q) = 0$, $\mathbb{Z}^\times = \{\pm 1\}$ and Φ_Q: $\{\pm 1\} \to H^0(\mathbb{Z}_p^\times) = \mathbb{F}_p^\times/\mathbb{F}_p^{\times 2}$ is non-trivial if and only if $-1 \notin \mathbb{F}_p^2$ or $p \equiv 3(4)$. Similarly Φ_Q^*: $\{\pm 1\} \to H^0(\mathbb{Z}_p^\times) \times \{\pm 1\}$ is injective. The result follows.

In the case $S\ell_2(\mathbb{F}_p)$, $E = Q(\zeta_p)^{\mathbb{Z}/2^{\ell-1}}$. If $p \equiv 3(4)$ then $\ell = 1$ and E is complex, so torsion vanishes. If $p \equiv 1(4)$ then the invariant part of E is the unique quadratic subfield $Q(\sqrt{p})$ of E. It is classically known that $\Gamma(E)$ had odd order and the fundamental unit E has norm -1 [Ha, Satz 7]. Then Φ^*: $A^\times/A^{\times 2} \to (\mathbb{F}_p^\times/\mathbb{F}_p^{\times 2}) \times \{\pm 1\} \times \{\pm 1\}$ is injective since $A^\times$ is generated by -1 and ε, mapping to $(?,-1,-1)$ and $(?,1,-1)$. The map Φ: $A^\times/A^{\times 2} \to \mathbb{F}_p^\times/\mathbb{F}_p^{\times 2}$ takes -1 to 1 since $-1 \in \mathbb{F}_p^2$ when $p = 4n+1$. As $N(\varepsilon) = -1$ the reduction $\bar{\varepsilon} \in \mathbb{F}_p$ satisfies $\bar{\varepsilon}^2 = -1$ and has therefore order 4 in $\mathbb{F}_p^\times = \mathbb{Z}/4n$. It is a square if and only if n is even, i.e. $p \equiv 1(8)$. //

The value of $L_n^Y(\mathbb{Z}\tau_2)$ and $L_n^Y(\mathbb{Z}\tau_3)$ depends on class numbers of the fields $Q(\zeta_d+\zeta_d^{-1})$ and on the maps Φ, Φ^* in 5.16 and it is difficult to give general concrete results. We note that

$SK_1(\mathbb{Z}\pi) = 0$ for $\pi = S\ell(\mathbb{F}_p), T\ell_2(\mathbb{F}_p)$, so the simple surgery obstruction groups $L^s_n(\pi)$ are equal to $L'_n(\pi)$, where we recall that $L'_{2n}(\pi) = L^Y_{2n}(\mathbb{Z}\pi)$ and $L'_{2n+1}(\pi) = L^Y_{2n+1}(\mathbb{Z}\pi)/(\mathbb{Z}/2)$.

Table 6.12: $L^s_n(S\ell_2(\mathbb{F}_p))$, $p < 29$

p	L^s_0	L^s_1	L^s_2	L^s_3
3	$(8\mathbb{Z})^2 \oplus (4\mathbb{Z})^3$	$\mathbb{Z}/2 \tilde{\times} \mathbb{Z}/2$	$(4\mathbb{Z})^2 \oplus \mathbb{Z}/2$	0
5	$(8\mathbb{Z})^4 \oplus (4\mathbb{Z})^5$	$(\mathbb{Z}/2)^3 \tilde{\times} (\mathbb{Z}/2)^5$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^3$
7	$(8\mathbb{Z})^4 \oplus (4\mathbb{Z})^5$	$(\mathbb{Z}/2)^2 \tilde{\times} (\mathbb{Z}/2)^3$	$(4\mathbb{Z})^2 \oplus (\mathbb{Z}/2)^2$	$\mathbb{Z}/2$
11	$(8\mathbb{Z})^5 \oplus (4\mathbb{Z})^8$	$(\mathbb{Z}/2)^4 \tilde{\times} (\mathbb{Z}/2)^5$	$(4\mathbb{Z})^2 \oplus (\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^4$
13	$(8\mathbb{Z})^6 \oplus (4\mathbb{Z})^{11}$	$(\mathbb{Z}/2)^5 \tilde{\times} (\mathbb{Z}/2)^{10}$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^7$
17	$(8\mathbb{Z})^7 \oplus (4\mathbb{Z})^{14} \oplus \mathbb{Z}/2$	$(\mathbb{Z}/2)^4 \tilde{\times} (\mathbb{Z}/2)^{15}$	$(\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^9$
19	$(8\mathbb{Z})^6 \oplus (4\mathbb{Z})^{15}$	$(\mathbb{Z}/2)^5 \tilde{\times} (\mathbb{Z}/2)^{12}$	$(4\mathbb{Z})^2 \oplus (\mathbb{Z}/2)^4$	$(\mathbb{Z}/2)^8$
23	$(8\mathbb{Z})^7 \oplus (4\mathbb{Z})^{18}$	$(\mathbb{Z}/2)^5 \tilde{\times} (\mathbb{Z}/2)^{14}$	$(4\mathbb{Z})^2 \oplus (\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^9$
29	$(8\mathbb{Z})^8 \oplus (4\mathbb{Z})^{25}$	$(\mathbb{Z}/2)^8 \tilde{\times} (\mathbb{Z}/2)^{23}$	$(\mathbb{Z}/2)^8$	$\mathbb{Z}/2 \tilde{\times} (\mathbb{Z}/2)^{16}$

where as above $(\mathbb{Z}/2)^m \tilde{\times} (\mathbb{Z}/2)^m$ is an unspecified extension L,

$$1 \to (\mathbb{Z}/2)^m \to L \to (\mathbb{Z}/2)^n \to 1.$$

Appendix. The proof of Theorem 6.7.

The calculation of $L^Y_i(\mathbb{Z}[Q2^k])$ (cf. [W6, §5]) is similar to the above calculation of $L^X_i(\mathbb{Z}[Q2^k n])(d)$, $d > 1$. but the results become more explicit, mainly because the strict class group has odd order for all the fields occuring.

Let $E_i = Q(\eta_i + \eta_i^{-1})$ with integers $A_i \subset E_i$, where η_i is a primitive 2^i'th root of 1. The center of the rational group ring of $Q2^k$ is equal to $\prod_{i=2}^{k-1} E_i \times E_0^4$; the simple component corresponding to E_{k-1} has type Sp, the other components have type O.

We briefly recall the calculation of $L^Y_i(\mathbb{Z}[Q2^k])$ in a form suitable for our purpose (cf. [W6, §5]). From 4.4 and 4.6,

A1

$$\text{Ker}\gamma_i:\ \Sigma;\ 0;\ 0;\ 0 \quad \text{for } i = 0,1,2,3$$

$$\text{Coker}\gamma_i:\ 0;\ (k+1)\cdot\mathbb{Z}/2;\ \mathbb{Z}/2;\ H^0(A^\times_{k-1,2}) \quad \text{for } i=0,1,2,3$$

where we have used that $\text{Coker}(H^0(A^\times) \to H^0(A_2^\times)) = \mathbb{Z}/2$ in the description of $\text{Coker}\gamma_1$.

Next we give a description of $H^i(Wh(\mathbb{Z}_2[Q2^k]))$ where $Wh(\mathbb{Z}_2\pi) = K_1(\mathbb{Z}_2\pi)/(\mathbb{Z}_2^\times \oplus \pi/\pi')$. Since $SK_1 = 0$ in our case, we can use reduced norms. Let $D2^k$ be the dihedral group of order 2^k.

<u>Proposition A2</u>. (i) $H^1(Wh(\mathbb{Z}_2[Q2^k])) = H^1(Wh(\mathbb{Z}_2[D2^k])) = 0$

ii) $H^0(Wh(\mathbb{Z}_2[Q2^k])) = \bigoplus_{i=2}^{k-1} H^0(1+2\pi_i A_i^\times) \oplus H^0(Wh(\mathbb{Z}_2[D4]))$ and $H^0(1+2\pi_i A_i^\times) = 2^{i-2}\cdot(\mathbb{Z}/2)$, $H^0(Wh(\mathbb{Z}_2[D4])) = 3\cdot\mathbb{Z}/2$.

(iii) The elements $1+(1-X^{2^{i-1}})X^j \in \mathbb{Z}_2[Q2^k]^\times$ $j = 1,\dots,2^{i-2}$ map trivially to $H^0(1+2\pi_r A_r^\times)$ for $r < i$ and generate $H^0(1+2\pi_i A_i^\times)$. The elements $1+2(1-X)$, $1+2(1-Y)$ and $1+(1-X)(1-Y)$ generate

$H^0(Wh(\mathbb{Z}_2[D4]))$.

<u>Proof</u>. (i) is proved in [W7], so we begin with (ii). There are exact sequences (cf. 2.5)

$$1+J_{k-1} \to Wh(\mathbb{Z}_2[Q2^k]) \to Wh(\mathbb{Z}_2[D2^{k-1}]) \to 0$$

$$1+J_i \to Wh(\mathbb{Z}_2[D^{i+1}]) \to Wh(\mathbb{Z}_2[D^{2^i}]) \to 0$$

where $1+J_{k-1} \cong 1+2\mathbb{Z}_2[\eta_{k-1}]^t[Y|Y^2=-1]$ and $1+J_i \cong 1+2\mathbb{Z}_2[\eta_i]^t[Y|Y^2=1]$ for $i \leq k-2$.

The reduced norm maps $1+J_i$ into $E^{\times}_{i,2}$. Moreover, using 2.7 on the cyclic algebra $(Q_2(\eta_i)/E_{i,2}, Y, \pm 1)$ we see that $Nrd(a_0+a_1Y) = N(a_0) \pm N(a_1)$ where $N: Q_2(\eta_i) \to E_{i,2}$ is the usual norm. Since $N(1+2\mathbb{Z}_2[\eta_i]) = 1+2\pi_i A^{\times}_{i,2}$ (cf.[S1]), $Nrd(1+J_i) = 1+2\pi_i A^{\times}_{i,2}$. We get exact sequences

$$1 \to 1+2\pi_{k-1}A^{\times}_{k-1,2} \to Wh(\mathbb{Z}_2[Q2^k]) \to Wh(\mathbb{Z}_2[D2^{k-1}]) \to 0$$

$$1 \to 1+2\pi_i A^{\times}_{i,2} \to Wh(\mathbb{Z}_2[D2^{i+1}]) \to Wh(\mathbb{Z}_2[D2^i]) \to 0$$

inducing short exact sequences of H^0-groups (since $H^1(Wh(\mathbb{Z}_2[D2^k]) = 0)$. We have $\log: 1+2\pi_i A^{\times}_{i,2} \cong 2\pi_i A^{+}_{i,2} \cong A^{+}_{i,2}$ and $A_{i,2}/2A_{i,2}$ is generated by $1+\pi^j$, $j=1,\dots,e$ where $e = 2^{i-2}$ is the remification index. This prove (ii).

Let $Nrd_i: \mathbb{Z}_2[Q2^k]^{\times} \to E^{\times}_i$ be the component of Nrd onto $E^{\times}_i$ in the product $\Pi E^{\times}_i$. It maps $1+(1-X^{2^{i-1}})X^j$ to $N(1+2\eta^j_i) = 5+2(\eta^j_i+\eta^{-j}_i)$ and for $r < i$ $Nrd_r(1+(1-X^{2^{i-1}})X^j) = 0$. Let $\pi_i = 2-\eta_i-\eta^{-1}_i$ be the prime element in $A_{i,2}$. We have $1+2(\eta^j_i+\eta^{-j}_i) \equiv \sum_{r \leq j} a_r \pi^r_i$ (modulo $1+4A_{i,2}$) with $a_j = 1$, and the first part of (iii) follows. The second part is similar. //

The Rothenberg sequence 2.1 relating $L^Y_i(\mathbb{Z}_2[Q2^k])$ and $L^K_i(\mathbb{Z}_2[Q2^k]) = \mathbb{Z}/2$ follows from [W3, Theorem 11] and gives

(A3) $L^Y_i(\mathbb{Z}_2[Q2^k]) = 0;\ L^K_1 \oplus H^0 \oplus \langle 5\rangle; \mathbb{Z}/2;\ L^K_3 \oplus H^0$ for $i = 0;1;2;3$

where H^0 is listed in A2 and $L^K_i = \mathbb{Z}/2$. (Note that $K_1(\mathbb{Z}_2[Q8])/Y = Wh(\mathbb{Z}_2[Q8]) \oplus \mathbb{Z}^{\times}_2/\langle -1\rangle$).

Finally, arguing as in 3.19 or from [W6, p.67-69] $\bar{\psi}_i: L^Y_i(\mathbb{Z}_2 Q[2^k]) \to \mathrm{Coker}\gamma_i$ is onto for i odd and zero for i even, so

(A4) $L^Y_i(\mathbb{Z}[Q2^k]) = \Sigma;\ \mathbb{Z}/2 \times [(2^{k-2}-k+3)\mathbb{Z}/2];\ \mathbb{Z}/2;\ (2^{k-3}+2)\mathbb{Z}/2$

for $i = 0;1;2;3$ (cf. [W6, Theorem 5.2.4]).

Let $\mathbb{Z}/3$ act on $Q8$ by cyclic permutation of X, Y and XY.

<u>Lemma A5</u>. $L^Y_1(\mathbb{Z}[Q8])^{\mathbb{Z}/3} = L^Y_1(\mathbb{Z}[Q8]);\ L^Y_3(\mathbb{Z}[Q8])^{\mathbb{Z}/3} = \mathbb{Z}/2$.

<u>Proof</u>. The elements

(*) $$1+(1-X^2)X,\quad 1+(1-X)(1-Y),\quad 1+2(1-X),\quad 1+2(1-Y)$$

in $\mathbb{Z}_2[Q8]$ generate $H^0 = H^0(Wh(\mathbb{Z}_2[Q8]))$.

We have the exact sequences

(A6) $$0 \to \mathrm{Coker}\gamma_{i+1} \to L^Y_i(\mathbb{Z}[Q8]) \to \mathrm{Ker}\bar{\psi}_i \to 0$$

and since $\bar{\psi}_i$ is onto, it follows from (A1) that $\mathrm{Ker}\bar{\psi}_1 = L^K_1 \oplus \mathbb{Z}/2$ and $\mathrm{Ker}\bar{\psi}_3 = L^K_3 \oplus 2\cdot(\mathbb{Z}/2)$.

The non-trivial element of L^K_i is given by the standard automorphism of the hyperbolic plane, and lifts to an invariant element of L^Y_i. Thus $\mathbb{Z}/3$ acts trivially on $\mathrm{Ker}\bar{\psi}_1$, and, since $\mathrm{coker}\gamma_2 = \mathbb{Z}/2$ also on $L^Y_1(\mathbb{Z}[Q8])$.

For $i=3$,

$$\bar{\psi}_3 : L_3^Y(\mathbb{Z}_2[Q8]) = L_3^K \oplus H^0 \to H^0(\mathbb{Z}_2^{\times})$$

is trivial on L_3^K and equal to Nrd_2 on H^0. The kernel of Nrd_2 is $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ generated by $(1+(1-X^2)X)(1+2(1-X))$ and $(1+(1-X^2)X)(1+2(1-Y))$. These element are not invariant, so $L_3^Y(\mathbb{Z}[Q8])^{\mathbb{Z}/3} \cong (\mathrm{Ker}\bar{\psi}_3)^{\mathbb{Z}/3} \cong L_3^K$. //

There are two conjugacy classes of subgroups $Q8$ in $Q2^k$. We have

<u>Proposition A7</u>. Each restriction $R_j^*: L_3^Y(\mathbb{Z}[Q2^k]) \to L_3^Y(\mathbb{Z}[Q8])$ has rank 2 for $k=4$ and rank 1 for $k \geq 5$, $L_3^K \subseteq L_3^Y(\mathbb{Z}[Q8])$ is not in the image and $\mathrm{Ker}\, R_1^* \neq \mathrm{Ker}\, R_2^*$.

<u>Proof</u>. Write $L_3^Y(\mathbb{Z}[Q2^k]) = L_3'(Q2^k) \oplus L_3^K$. Then

(A8) $L_3'(Q2^k) = \mathrm{Ker}\{H^0(Wh(\mathbb{Z}_2[Q2^k])) \xrightarrow{Nrd_{k-1}} H^0(A_{k-1,2}^{\times})\}$

The summand L_3^K is mapped trivially to $Q8$, so we can concentrate on $Res: L_3'(Q2^k) \to L_3'(Q8)$.

We begin with a general remark. Suppose $H \subset G$ is a subgroup of index 2 and let $T \in G \setminus H$. Each element in $\mathbb{Z}_2 G$ can be presented by $\alpha_0 + \alpha_1 T$ with $\alpha_0, \alpha_1 \in \mathbb{Z}_2 H$. The map

$$(\mathbb{Z}_2 G)^{\times} \xrightarrow{\det} K_1(\mathbb{Z}_2 G) \xrightarrow{Res} K_1(\mathbb{Z}_2 H)$$

is calculated as

$$Res(\alpha_0+\alpha_1 T) = \det\begin{pmatrix}\alpha_0 & \bar{\alpha}_1 T^2\\ \alpha_1 & \bar{\alpha}_0\end{pmatrix}$$

$$= \det\begin{pmatrix}1 & 0\\ -\alpha_1\alpha_0^{-1} & 1\end{pmatrix}\det\begin{pmatrix}\alpha_0 & \bar{\alpha}_1 T^2\\ \alpha_1 & \bar{\alpha}_0\end{pmatrix}$$

$$= \det(\alpha_0\bar{\alpha}_0 - \alpha_0\alpha_1\alpha_0^{-1}\bar{\alpha}_1 T^2)$$

Consider the subgroups $Q8 \subset Q16 \subset \ldots \subset Q2^k$ given by $\langle X^{2^i}, Y\rangle \subset \langle X,Y\rangle = Q2^k$. We calculate

$$Res^{k-3}: H^0(Wh(\mathbb{Z}_2[Q2^k])) \xrightarrow{Res} H^0(Wh(\mathbb{Z}_2[Q2^{k-1}])) \xrightarrow{Res} \ldots$$
$$\ldots \xrightarrow{Res} H^0(Wh(\mathbb{Z}_2[Q8]))$$

on the generators from A2(iii). The above formula gives

$$Res(1+(1-X^{2^{i-1}})X^j) = (1+(1-X^{2^{i-1}})X^j)^2 = 0, \quad j \text{ even}$$
$$= (1+(1-X^{2^{i-1}})^2 X^{2j}, \quad j \text{ odd}$$

and hence

(A9) $$Res^{k-3}(1+(1-X^{2^{i-1}})X^j) = (1+(1-X^m)^2 X^j)(1+4r(X))$$

where j is odd, $i>2$, $m=2^{k+i-5}$ and $r(X) \in \mathbb{Z}_2[X]$ has $r(1)=r(-1)=0$.

(A10)
$$Res^{k-3}(1+2(1-X)) = 3^{4n} - (2X)^{2n}$$
$$Res^{k-3}(1+2(1-Y)) = (1+2(1-Y))(1+2(1-X^2Y))(9-4Y^2)^{n-1}$$

with $n = 2^{k-4}-1$.

The reduced norm

$$Nrd_{k-1}: H^0(Wh(\mathbb{Z}_2[Q2^k])) \to H^0(A_{k-1,2}^{\times})$$

takes the values

$$Nrd_{k-1}(1+2(1-X)) = 5+2\pi_{k-1} \in 1+2\pi_{k-1}A^{\times}_{k-1,2}$$

$$Nrd_{k-1}(1+2(1-Y)) = 5 \qquad \in 1+2\pi_{k-1}A^{\times}_{k-1,2}$$

$$Nrd_{k-1}(1+(1-X)(1-Y)) = 1+\pi_{k-1},\ Nrd_{k-1}(1+(1-X^2)X) = 1+\pi^3_{k-1}$$

If $k > 4$, then the cosets of $1+(1-X)(1-Y)$ and $1+(1-X^2)X$ modulo $Ker(Res^{k-3})$ do not intersect $Ker(Nrd_{k-1})$, so these elements do not define elements in $L_3'(Q2^k)$. On the other hand, it is easy to correct $1+2(1-X)$ and $1-2(1-Y)$ to elements in $L_3'(Q2^k))$, cf. (A2). Finally, the injection

$$Nrd: H^0(Wh(\mathbb{Z}_2[Q8])) \to H^0(1+4\mathbb{Z}_2^{\times}) \oplus 3\cdot H^0(\mathbb{Z}_2^{\times}/<\pm 1>)$$

shows that $Res^{k-3}(1+2(1-Y)) \neq 0$ but $Res^{k-3}(1+2(1-X)) = 0$. So $Res: L_3'(Q2^k) \to L_3'(Q8)$ has rank 1.

If $k = 4$, then $[1+(1-X^2)X][1+2(1-X)]$ and $[1+(1-X^2)X][1+(1-Y)]$ represent elements of $L_3'(Q16)$.

The other conjugacy class $Q8 \subset Q2^k$ generated by $X^{2^{k-3}}$ and XY is treated similarly. The elements containing only X have restrictions as before, but $1+2(1-Y)$ restricts to $9-4Y^2$ in $Q2^{k-1}$ and to 0 in $H^0(Wh(\mathbb{Z}_2[Q2^{k-2}]))$. Only the element $1+2(1-X) \in$ belongs to the kernels of both restrictions. //

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